CHAPTER 1

# Introduction

Our aim in this chapter is to describe informally a variety of concrete examples that show why stacks are needed, and to illustrate some of the key ingredients of stacks. We start with a brief discussion of the two natures of a stack: as categories, and from atlases/groupoids. In proctise it is usually easy to define the appropriate category, but it requires some work, requiring knowledge of the geometry involved, to construct an atlas. Then we look at examples, where these and other "stacky" features can be seen. Many of these examples should be familiar to the reader in some setting. Some of them were important in the early history of stacks, so reading about them will also give a glimpse of this history. Most of these examples will reappear later in the book, and most of the ideas seen here will be developed systematically later. Depending on a reader's background, statements made without proof can be accepted as facts to be used for motivation, or proofs can be worked out as exercises.

Making the notion of stack precise requires a fair amount of rather abstract language, including such mouthfuls as "categories fibered in groupoids". Starting in the next chapter we will develop this language slowly and carefully, with precise versions of most of these and many other examples. We hope that seeing several examples will help the reader digest what is to follow. However, we emphasize that nothing that is done here is logically necessary for reading the rest of the book.

## 1. Stacks as categories

Stacks are defined with respect to some fixed category $\mathcal{S}$, called the *base category.* For example, $\mathcal{S}$ can be the category (Sch) of schemes (either all schemes or schemes over a fixed base), or $(\mathbb{C}_{\mathrm{an}})$ of complex analytic spaces, or (Diff) of differentiable manifolds, or (Top) topological spaces, or even the category (Set) of sets. A *stack* over $\mathcal{S}$ will be a category $\mathcal{X}$ together with a functor $\mathcal{X} \to \mathcal{S}$, satisfying some properties — most of which will be left until later to discuss. These properties will depend, in part, on a "topology" on $\mathcal{S}$. A *morphism* from one $\mathcal{X} \to \mathcal{S}$ to another $\mathcal{Y} \to \mathcal{S}$ is defined to be a functor from $\mathcal{X}$ to $\mathcal{Y}$ that commutes with the projections to $\mathcal{S}$.

We start with some examples of this. (If definitions here are too sketchy, details can be found in Chapter 2.)

EXAMPLE 1.1A. **Objects (Schemes).** An object $X$ in $\mathcal{S}$ determines a category $\mathcal{X}$, whose objects are pairs $(S, f)$, where $S$ is an object in $\mathcal{S}$ and $f\colon S \to X$ is a morphism. A morphism from $(S', f')$ to $(S, f)$ in $\mathcal{X}$ is given by a morphism $g\colon S' \to S$ such that $f \circ g = f'$. The functor $\mathcal{X} \to \mathcal{S}$ takes an object $(S, f)$ to $S$, and takes a morphism from $(S', f')$ to $(S, f)$ to the underlying morphism from $S'$ to $S$. It is a basic fact of

Grothendieck/Yoneda that this category $\mathcal{X}$ determines $X$ up to canonical isomorphism. This category is often denoted $\underline{X}$; when the idea of stacks has been thoroughly digested, it can be denoted simply by $X$, but we will not do this in Part I. We will be primarily interested in the case when $X$ is a scheme, and $\mathcal{S}$ is a category of schemes, but the notion is valid for any base category $\mathcal{S}$.

This example is a variation of Grothendieck's idea of replacing a scheme $X$ by its functor of points. This is the functor $h_X$ from $\mathcal{S}$ to sets, with $h_X(S) = \mathrm{Hom}(S, X)$, called the set of $S$-*valued points* of $X$. A morphism $f: X \to Y$ determines a natural transformation from $h_X$ to $h_Y$, taking $g: S \to X$ to $f \circ g: S \to Y$. Just as $X$ can be recovered from $h_X$, the $f$ can be recovered from the natural transformation $h_X \to h_Y$.

EXAMPLE 1.1B. **Moduli of curves.** Let $\mathcal{S}$ be the category of schemes, or schemes over a fixed base. A family of curves of genus $g$ is a morphism $C \to S$ of schemes which is smooth and proper, whose geometric fibers are connected curves of genus $g$. The moduli stack $\mathcal{M}_g$ of curves of genus $g$ has for its objects such families. A morphism from a family $C' \to S'$ to $C \to S$ is pair of morphisms $C' \to C$ and $S' \to S$ such that the diagram

$$\begin{array}{ccc} C' & \longrightarrow & C \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

is cartesian, i.e., it is commutative and the induced map from $C'$ to the fibered product $S' \times_S C$ is an isomorphism. It is important to note that the morphism from $S'$ to $S$ does *not* determine the morphism from $C'$ to $C$; for example, if $S$ and $S'$ are points, and $C' = C$, an automorphism of $C$ will be a nontrivial morphism in $\mathcal{M}_g$. The functor $\mathcal{M}_g \to \mathcal{S}$ is the obvious one, as it is with most moduli problems: it takes an object $C \to S$ of $\mathcal{M}_g$ to the object $S$ of $\mathcal{S}$, and a morphism from $C' \to S'$ to $C \to S$ to the underlying morphism from $S'$ to $S$.

To know all about the algebraic geometry of curves and their automorphisms is to know all about $\mathcal{M}_g \to \mathcal{S}$. The existence of nontrivial automorphisms is what prevents $\mathcal{M}_g$ from "being" a scheme.

A more classical object, the "coarse" moduli space $M_g$, which is a quasiprojective scheme over the integers, was constructed in [**75**], cf. [**54**], [**55**], [**29**]. Its geometric points correspond to isomorphism classes of curves, and it has the property that for any family of curves $C \to S$, there is a canonical morphism from $S$ to $M_g$ taking a geometric point $s$ to the isomorphism class of the fiber $C_s$. These morphisms determine a functor from $\mathcal{M}_g$ to the category $\underline{M}_g$ determined by $M_g$. Moreover, $M_g$ is characterized by a universal property that can be described as follows. Any morphism from $\mathcal{M}_g$ to the category $\underline{N}$ of a scheme $N$ must factor uniquely through $\underline{M}_g$ (cf. [**75**], §5.2); note that the morphism from $\underline{M}_g$ to $\underline{N}$ is given by a morphism from $M_g$ to $N$. In the language of stacks, the space $M_g$ will be a "coarse moduli space" for the stack $\mathcal{M}_g$.

The scheme $M_g$ is related to the functor from the category (Sch) of schemes to the category (Set) of sets, that takes a scheme $S$ to the set of *isomorphism classes* of curves of genus $g$ over $S$. Unlike this functor, the category $\mathcal{M}_g$ does not identify isomorphic families.

Many important examples of stacks will be variations of this example. For example, there is a stack $\mathcal{M}_{g,n}$ whose objects are families of curves $C \to S$ together with $n$ pairwise disjoint sections $\sigma_1, \ldots, \sigma_n$ from $S$ to $C$; the morphisms in $\mathcal{M}_{g,n}$ must be compatible with these sections. There are also "compactifications", which allow fibers to be nodal curves, with an appropriate notion of stability. One can also replace curves by other varieties.

The use of stacks in [**20**] to prove the irreducibility of the variety $M_g(k)$ of curves of genus $g$ over any algebraically closed field $k$ can be sketched as follows. Take $\mathcal{S}$ to be all schemes. Suppose $\mathcal{M}_g$ were represented by a scheme $M_g$ that is smooth over $\mathrm{Spec}(\mathbb{Z})$, and that $\mathcal{M}_g$ had a compactification $\overline{\mathcal{M}_g}$ (using stable curves) that is represented by a scheme $\overline{M}_g$ that contains $M_g$ as an open subscheme, with $\overline{M}_g$ smooth and projective over $\mathrm{Spec}(\mathbb{Z})$. The classical fact that $\overline{M}_g(\mathbb{C})$ is connected would imply, by a connectedness theorem of Enriques and Zariski, that all geometric fibers $\overline{M}_g(k)$ of $\overline{M}_g$ over $\mathrm{Spec}(\mathbb{Z})$ are connected. Since a nonsingular connected variety is irreducible, the open subvariety $M_g(k)$ would also be irreducible. Although these assertions are all false for the coarse moduli spaces — even $M_g(\mathbb{C})$ is singular — they are true, suitably interpreted, for the corresponding stacks, and the irreducibility of the coarse varieties $M_g(k)$ follows.

EXAMPLE 1.1C. **Torsors.** Here we start with $\mathcal{S} = (\mathrm{Top})$ being the category of topological spaces. Let $G$ a topological group. A *$G$-torsor*, or a *principal $G$-bundle*, is a (continuous) map $E \to S$, with a (continuous) action of $G$ on $E$, which we take to be a right action; one requires that it be locally trivial, in the sense that $S$ has an open covering $\{S_\alpha\}$ such that the restriction $E|_{S_\alpha}$ is isomorphic to the trivial bundle $S_\alpha \times G \to S_\alpha$. One has a category, which we denote by $BG$, whose objects are the $G$-torsors $E \to S$. (We will explain this notation later.) A morphism from $E' \to S'$ to $E \to S$ is given pair of maps $E' \to E$ and $S' \to S$ such that the map $E' \to E$ is equivariant (commutes with the actions of $G$), and the induced diagram is cartesian as with the case of curves. The functor from $BG$ to $\mathcal{S}$ is defined similarly as well.

There is an important generalization of this example. If $G$ acts on the right on a space $X$, one defines a category, denoted $[X/G]$, whose objects are $G$-torsors $E \to S$, together with an equivariant map from $E$ to $X$. A morphism from $E' \to S', E' \to X$ to $E \to S, E \to X$ is given pair of maps $E' \to E$ and $S' \to S$ giving a map of torsors as above, but, in addition, so that the composite $E' \to E \to X$ is equal to the given map from $E'$ to $X$. This may look rather arbitrary now, but we will soon see examples where these categories arise naturally; we will see later why $[X/G]$ serves as the "right" quotient of $X$ by the action of $G$. Note that the category $BG$ is the same as the category $[\bullet/G]$, where $\bullet$ is a point; and $[X/\{1\}]$ (where $\{1\}$ denotes the group with one element) is the same as $\underline{X}$. If $G$ acts on the left on $X$, and we consider left $G$-torsors, we have similarly a category denoted $[G\backslash X]$.

Both of these examples extend to the case where $\mathcal{S}$ is a category of schemes over a fixed base, with algebraic actions of algebraic groups (group schemes). The only difference is that the notion of local triviality for a torsor is usually taken, not in the Zariski topology, but in the étale topology.

A *morphism* from a stack $\mathcal{X}$ to a stack $\mathcal{Y}$ over $\mathcal{S}$ is simply a functor from $\mathcal{X}$ to $\mathcal{Y}$ that commutes with the projections to $\mathcal{S}$. However, the natural notion of isomorphism in the world of categories is not a strict isomorphism (which would be bijective on objects and morphisms), but is an *equivalence* of categories; a morphism which is an equivalence of categories will be an isomorphism of stacks. So quite different looking categories can give rise to isomorphic stacks. Another complicatgion with the categorical notion is that it is not easy to do algebraic geometry on a category! For example, we would like to say that $\mathcal{M}_g$ is smooth, and that it is an open substack of a smooth compactification, with complement having normally crossing divisors. We would also like to describe line bundles and vector bundles on these stacks, do intersection theory on them, and compute cohomology groups.

REMARK 1.2. The notion of $S$-valued points, discussed in Example 1.1A, can be used to make casual set-theoretic notation rigorous. For example, if $\mathcal{S}$ is the category $((\mathrm{sch})/\Lambda)$ of schemes over a base $\Lambda$, and a group scheme $G$ over $\Lambda$ acts on the right on a scheme $X$ in $\mathcal{S}$, the associativity condition "$(x \cdot g) \cdot h = x \cdot (g \cdot h)$" is not strong enough if applied only to classical (geometric) points, but it is if applied to $S$-valued points for all $S$ in $\mathcal{S}$. Here $x$, $g$, and $h$ are taken to be in $h_X(S)$, $h_G(S)$, and $h_G(S)$, and $x \cdot g$ denotes the composite $S \xrightarrow{(x,g)} X \times_\Lambda G \xrightarrow{\sigma} X$, where $\sigma$ is the action. The equation "$(x \cdot g) \cdot h = x \cdot (g \cdot h)$" for all such $x$, $g$, and $h$ is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} X \times \Lambda G \times \Lambda G & \xrightarrow{\sigma \times \Lambda 1_G} & X \times \Lambda G \\ {\scriptstyle 1_X \times m}\downarrow & & \downarrow{\scriptstyle \sigma} \\ X \times \Lambda G & \xrightarrow[\sigma]{} & X \end{array}$$

where $m \colon G \times \Lambda G \to G$ is the product in $G$; one sees this by considering the universal case where $S$ is $X \times \Lambda G \times \Lambda G$, with $x$, $g$, and $h$ the three projections. We will often use such abbreviations in this text.

## 2. Stacks from groupoids

An algebraic stack will come from a kind of atlas, which is called a groupoid. If $\mathcal{S}$ is the base category, a *groupoid in* $\mathcal{S}$, or an $\mathcal{S}$-*groupoid*, consists of a pair of objects $U$ and $R$ in $\mathcal{S}$, together with five morphisms: $s$ (the "source") and $t$ (the "target") from $R$ to $U$, $e$ (the "identity") from $U$ to $R$, a morphism $m$ (the "multiplication") from the fibered product[1] $R\ {}_t{\times}_{U,s} R$ to $R$, and a morphism $i$ (the "inverse") from $R$ to $R$, which satisfy some natural axioms.

[1]For this to make sense, we assume, at least for now, that the fibered product of $R$ with itself over $U$, using the two projections $t$ and $s$, must exist in the category $\mathcal{S}$; we usually abbreviate this to $R\ {}_t{\times}_s R$. Similarly whenever we write cartesian products such as $U \times U$, we are assuming they exist as well.

In fact, you already know how to write down these axioms, as follows. Take a category in which all maps are isomorphisms, and let $U$ be the objects of this category, and $R$ the morphisms or arrows, with $s$ and $t$ be the usual source and target, $e$ the identity (taking an object to the identity map on it), $m$ the composition (taking a pair $f \times g$ to $g \circ f$), and $i$ the inverse. The axioms for a category amount to certain compatibilities among these morphisms, such as $s \circ e = \mathrm{id}_U$. Writing down these compatibilities amounts exactly to writing down the axioms for a groupoid. It is an excellent exercise to do this now, checking with Chapter 4 to see if you have missed any.

A notation like $(U, R, s, t, e, m, i)$ for a groupoid is too unwieldy to be practical. We will often use the notation $R \rightrightarrows U$ for a groupoid with spaces $U$ and $R$ indicated, with the two arrows (for $s$ and $t$) standing as an abbreviation for all five maps. In fact, $e$ and $i$ are uniquely determined by $s$, $t$, and $m$, so we often leave their construction to the reader. When $\mathcal{S} = (\mathrm{Top})$, we call this a *topological groupoid*, and when $\mathcal{S} = (\mathbb{C}_{\mathrm{an}})$, we call it an *analytic groupoid*; when $\mathcal{S}$ is a category of schemes, it will be called an *algebraic groupoid* or a *groupoid scheme*.

In these geometric settings, the *isotropy group* $\mathrm{Aut}(x)$ of a point $x$ in $U$ is the set $s^{-1}(x) \cap t^{-1}(x) \subset R$, which is a group with product determined by $m$.

A morphism from a groupoid $R' \rightrightarrows U'$ to a groupoid $R \rightrightarrows U$ is given by a pair $(\phi, \Phi)$ of morphisms $\phi\colon U' \to U$ and $\Phi\colon R' \to R$, commuting with all the morphisms.

One geometric example of a groupoid, called the *fundamental groupoid* of a topological space, is probably familiar to you. Although it will not play much of a role in this book, it shows clearly the not-everywhere-defined grouplike structure of a groupoid. If $X$ is a topological space, its fundamental groupoid can be denoted $\Pi(X) \rightrightarrows X$. The elements of $\Pi(X)$ are triples $(x, y, \sigma)$, with $x$ and $y$ points of $X$, and $\sigma$ a homotopy class of paths in $X$ starting at $x$ and ending at $y$; $s$ and $t$ take this triple to $x$ and $y$ respectively, and $m((x, y, \sigma), (y, z, \tau)) = (x, z, \sigma * \tau)$, where $\sigma * \tau$ is the usual product coming from tracing first a path representing $\sigma$ and then a path representing $\tau$.[2] This groupoid has advantages over the usual fundamental group (which requires an arbitrary choice of base point), particularly in the study of the Van Kampen theorem when the intersection of the open sets involved is not connected (cf. [**16**]). There are also useful variants of the fundamental groupoid, such as the groupoid $\Pi(X, A) \rightrightarrows A$, where $A$ is a subset of $X$, and the paths connect points of $A$. If $X$ is a foliated manifold, one can require the paths and homotopy equivalences to lie within leaves of the foliation; if one replaces homotopy equivalence by holonomy equivalence, one arrives at the *holonomy groupoid* of the foliation [**43**].

A continuous mapping $f\colon X \to Y$ determines a morphism $(f, F)$ from the groupoid $\Pi(X) \rightrightarrows X$ to the groupoid $\Pi(Y) \rightrightarrows Y$, with $F(\sigma) = f \circ \sigma$. A homotopy $H\colon X \times [0,1] \to Y$ from $f$ to $g$ determines a mapping $\theta\colon X \to \Pi(Y)$, taking $x$ in $X$ to the path $t \mapsto H(x, t)$. If likewise $(g, G)$ denotes the morphism of groupoids determined by $g$, this

[2] For a general space, its fundamental groupoid is a groupoid of sets. If $X$ has a universal covering space, i.e., $X$ is semilocally simply connected, then $\Pi(X)$ has a natural topology so that $s$ and $t$ are local homeomorphisms, and the fundamental groupoid is a topological groupoid.

mapping $\theta$ satisfies the identities

$$s(\theta(x)) = f(x), \quad t(\theta(x)) = g(x), \quad \text{and} \quad \theta(s(\sigma)) \cdot G(\sigma) = F(\sigma) \cdot \theta(t(\sigma))$$

for $x$ in $X$ and $\sigma$ in $\Pi(X)$. Maps $\theta$ satisfying these identities are called 2-*isomorphisms*; they will be the analogues of homotopies for groupoids.

EXAMPLE 1.3A. **Classical atlases.** If $X$ is a scheme, or a topological space, and $\{U_\alpha\}$ is an open covering of $X$ (with $\alpha$ varying in some index set), let $U = \coprod U_\alpha$ be the disjoint union, and let $R = \coprod U_\alpha \cap U_\beta$, the disjoint union of all intersections over all ordered pairs $(\alpha, \beta)$; equivalently, $R = U \times_X U$. The five maps are the obvious ones: $s$ takes a point in $U_\alpha \cap U_\beta$ to the same point in $U_\alpha$, and $t$ takes it to the same point in $U_\beta$; $e$ takes a point in $U_\alpha$ to the same point in $U_\alpha \cap U_\alpha$; $i$ takes a point in $U_\alpha \cap U_\beta$ to the same point in $U_\beta \cap U_\alpha$; for $m$, if $u$ is in $U_\alpha \cap U_\beta$ and $v$ is in $U_\delta \cap U_\gamma$, requiring $s(u)$ to equal $t(v)$ says that $\beta = \delta$ and $u = v$, so we can set $m(u, v) = u = v$ in $U_\alpha \cap U_\gamma$.

One of the basic constructions of algebraic geometry, known as gluing (recollement in French), amounts to constructing $X$ from a compatible collection of schemes $\{U_\alpha\}$, with isomorphisms from an open set $U_{\alpha\beta}$ of each $U_\alpha$ to an open set $U_{\beta\alpha}$ of $U_\beta$, satisfying axioms of compatibility. These axioms are the same as those for constructing a manifold by gluing open subsets of Euclidean spaces.

There is a similar atlas (groupoid) constructed from an étale covering $\{U_\alpha \to X\}$, but taking $R$ to be $\coprod U_\alpha \times_X U_\beta$. In fact, for any morphism $U \to X$, one can construct a groupoid, with $R = U \times_X U$, with $s$ and $t$ the two projections, $e$ the diagonal, $i$ the map reversing the two factors, and $m$ the composite

$$(U \times_X U) \times_U (U \times_X U) \cong U \times_X U \times_X U \to U \times_X U,$$

where the second map is the projection $p_{1,3}$ to the outside factors. Applying this to the case of an étale covering $U = \coprod U_\alpha \to X$ recovers the "gluing" atlas.

A trivial but important special case of this construction takes, for any object $X$ of our category $\mathcal{S}$, the groupoid arising from the identity map from $X$ to $X$. Here $U = X$, $R = X$, and all the maps of the groupoid are identity maps. When $\mathcal{S}$ is the category of sets, so a groupoid is identified with a category, a set is exactly a category in which the only maps are identity maps. In this sense, one may say that schemes (or spaces) are to stacks as sets are to (groupoid) categories.

In this collection of examples, the canonical map $(s, t)\colon R \to U \times U$ is an embedding (a monomorphism), so that $R$ defines an equivalence relation on $U$, and $X$ may be thought of as the quotient of $U$ by this equivalent relation. In fact, algebraic spaces are constructed from equivalence relations $R \to U \times U$ with projections $s$ and $t$ étale. (Any equivalence relation on a set $U$, in fact, determines a groupoid of sets.) One major difference between a scheme or algebraic space and a general stack is that, for an atlas for a stack, the morphism from $R$ to $U \times U$ need not be one-to-one (on geometric points).

EXAMPLE 1.3B. **Group actions.** Suppose an algebraic (resp. topological) group $G$ acts on a scheme (resp. topological space) $U$, say on the right. There is a natural equivalence relation on $U$: two points $u$ and $v$ are equivalent if they are in the same orbit: $v = u \cdot g$ for some $g \in G$. There is a better groupoid to construct from this action: take $R = U \times G$, and think of a point $(u, g)$ in $R$ as being a point $u$ together with an arrow $g$ from $u$ to $u \cdot g$. This suggests that we take $s\colon U \times G \to U$ to be the first projection and $t\colon U \times G \to U$ to be the action (so $s(u, g) = u$ and $t(u, g) = u \cdot g$); $e$ is the identity ($e(u) = (u, e_G)$), and

$$m\big((u, g), (u \cdot g, h)\big) = (u, g \cdot h),$$

and $i(u, g) = (u \cdot g, g^{-1})$.

EXERCISE 1.1. Verify that this $U \times G \rightrightarrows U$ satisfies the axioms to be a groupoid.

This groupoid is sometimes denoted by a semi-direct product notation $U \rtimes G$, and it is called a *transformation groupoid*. This groupoid will, in fact, be an atlas for the stack $[U/G]$ discussed in Example 1C. Note that for $x$ in $U$, the isotropy group $\mathrm{Aut}(x)$ of the groupoid is the same as the isotropy or stabilizer group $G_x$ for the group action. Whenever there is are fixed points, the mapping $(s, t)\colon R \to U \times U$ is not an embedding: if $u \in U$ and $g \in G$, with $g \neq e_G$ and $u \cdot g = u$, then $(u, g)$ and $(u, e_G)$ have the same image. The stack determined by this groupoid will capture the action better than the naive quotient $U/G$, when this latter quotient exists. An extreme example is the action of $G$ on a point $\bullet$; the groupoid $G \rightrightarrows \bullet$ carries the information of the group $G$ (and the stack $BG$ from Example 1.1C), but the quotient space is just the point $\bullet$.

Let us briefly compare the stack quotient with other notions of quotient. Here for simplicity the base category is taken to be varieties over the complex numbers. If an algebraic group $G$ acts on a variety $U$, a *categorical quotient* is a variety $U/G$, with a surjective morphism $q\colon U \to U/G$ that is constant on the orbits of $G$ in $U$, and satisfies the universal property: for any variety $Y$, and any morphism $f\colon U \to Y$ that is constant on $G$-orbits, there is a unique morphism $\bar{f}\colon U/G \to Y$ such that $f = \bar{f} \circ q$.

EXAMPLE 1.4. Let $U = \mathbb{C}^2$, $G = \mathbb{C}^*$, with $G$ acting on $U$ by the action $(x, y) \cdot t = (xt, yt)$. Let $U^\circ = U \smallsetminus \{(0, 0)\}$. The following are not hard to verify, and, as usual, the reader is invited to verify them:

- (1*a*) The canonical map from $U^\circ$ to $\mathbb{P}^1$ that takes $(x, y)$ to $[x : y]$ identifies $\mathbb{P}^1$ as the categorical quotient $U^\circ/G$.
- (1*b*) This map induces a bijection between $G$-orbits on $U^\circ$ and points of $\mathbb{P}^1$.
- (1*c*) If $S$ is any complex variety there is a bijection between maps $S \to \mathbb{P}^1$ and $G$-torsors $E \to S$ equipped with $G$-equivariant morphism $E \to U^\circ$, up to $G$-equivariant isomorphisms of $G$-torsors commuting with the maps to $U^\circ$.
- (2*a*) The categorical quotient $U/G$ is a point.
- (2*b*) There is, however, more than one $G$-orbit on $U$.
- (2*c*) A $G$-torsor $E \to S$ with equivariant map $E \to U$ is not determined by the corresponding map from $S$ to a point.

The assersions of (1c) show that on $U^\circ$, where $G$ acts freely, the classical quotient $U^\circ/G$ represents the quotient stack $[U^\circ/G]$. In the world of stacks, $[U/G]$ will contain $[U^\circ/G]$ as a dense open substack, as contrasted with the classical notion of categorical quotient, in which $U/G$ is a point; note that the set of $G$-orbits has no sensible topology, let alone a structure of an algebraic variety (or scheme).

An analogous groupoid $G \times U \rightrightarrows U$ arises from a left action of a group $G$ on $U$. This groupoid, also denoted $G \ltimes U$, is defined by setting $s(g,u) = u$, $t(g,u) = g \cdot u$, and $m((g,u),(g',g \cdot u)) = (g' \cdot g, u)$. More generally, if $G$ acts on the left on $U$, and $H$ acts on the right on $U$, and the actions commute, in the sense that $(g \cdot u) \cdot h = g \cdot (u \cdot h)$ for all $g \in G$, $u \in U$, and $h \in H$, there is a groupoid

$$G \times U \times H \rightrightarrows U,$$

with $s(g,u,h) = u$, $t(g,u,h) = g \cdot u \cdot h$, and $m((g,u,h),(g', g \cdot u \cdot h, h') = (g' \cdot g, u, h \cdot h')$. This groupoid may be denoted $G \ltimes U \rtimes H$.

Even when there is a sensible quotient variety, the groupoid $U \rtimes G$ is often better than the space $U/G$, much as the stack $\mathcal{M}_g$ is better than the coarse moduli space $M_g$. In fact, this is more than an analogy. The moduli space $M_g$ can be constructed [**75**] as the quotient of a Hilbert scheme Hilb by an action of a group $\mathrm{PGL}(N)$. We will see, in fact, that the groupoid $\mathrm{PGL}(N) \times \mathrm{Hilb} \rightrightarrows \mathrm{Hilb}$ is an atlas for $\mathcal{M}_g$.

As one would expect from the notion of manifolds, many different groupoids can be atlases for the same stack. Given atlases for manifolds need to be refined to be compatible with morphisms between manifolds. There are similar notions for atlases for stacks. Of course, an arbitrary map of groupoids $(\varphi, \Phi)$ from $R' \rightrightarrows U'$ to $R \rightrightarrows U$ will not determine an isomorphism of their corresponding stacks. There are two properties that will in fact guarantee this, the first corresponding to injectivity, the second to surjectivity. The properties are:

(i) *The diagram*

$$\begin{array}{ccc} R' & \xrightarrow{(s,t)} & U' \times U' \\ \Phi \downarrow & & \downarrow \varphi \times \varphi \\ R & \xrightarrow{(s,t)} & U \times U \end{array}$$

*must be cartesian.*

Note that this can be expressed in terms of $S$-valued points: the map $h_{R'}(S) \to h_R(S) \times_{h_{U \times U}(S)} h_{U' \times U'}(S)$ is a bijection for all $S$.

(ii) For every $u \in U$ there is a $u' \in U'$ and an $a \in R$ such that $s(a) = \varphi(u')$ and $t(a) = u$; in other words: *The morphism*

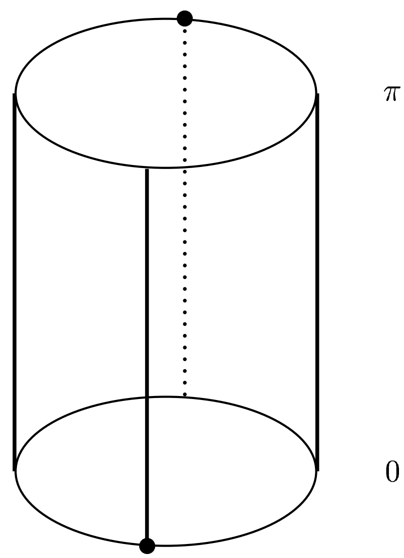
$$V = U' {}_{\varphi}\!\times_{U,s} R \longrightarrow U$$

*determined by $t$ must be surjective.*

Here, however, "surjective" must be interpreted correctly. Requiring surjectivity on the naive point level is too weak, and requiring that $h_V(S) \to h_U(S)$ be surjective for all $S$ is too strong, since that is equivalent to the existence of a splitting morphism from $U$ to $V$. (For example, a fiber bundle projection should be surjective, but it may have no global section.) What works is to require that the map must be locally surjective, using the topology on $\mathcal{S}$. That is, we require: *$U$ has a covering $\{U_\alpha \to U\}$ such that each $U_\alpha \to U$ factors through $V$.*

In the case where $\mathcal{S}$ is the category of sets (with the discrete topology), so the groupoids are categories and maps between them are functors, condition (i) says that this functor is fully faithful, and condition (ii) says that it is essentially surjective; together they say that the functor is an equivalence of categories.

Here is a geometric example. Let $D = \{z \in \mathbb{C} \mid |z| \le 1\}$, and let $X$ be the cylinder $D \times \mathbb{R}$, with the identification $(z, \phi) \sim (z', \phi')$ if $\phi' - \phi = n\pi$, $n \in \mathbb{Z}$, and $z' = (-1)^n z$.




The group $S^1$ acts on $X$, by $e^{i\vartheta} \cdot (z, \varphi) = (z, \vartheta + \varphi)$. (This is an example of a "Seifert circle bundle".) The group $\{\pm 1\}$ acts on $D$ by $(-1) \cdot z = -z$, and $\{\pm 1\}$ is a subgroup of $S^1$ by $-1 \mapsto e^{i\pi}$. The embedding $D \to X$, $z \mapsto (z, 0)$, is equivariant with respect to $\{\pm 1\} \to S^1$, giving a morphism of groupoids

$$\{\pm 1\} \ltimes D \;\to\; S^1 \ltimes X.$$

EXERCISE 1.2. (a) Show that this morphism satisfies properties (i) and (ii), where the base category $\mathcal{S}$ is (Top) or (Diff). (b) Compute the isotropy groups of these actions at all points.

One difficulty in learning about stacks is that it is not very easy to "translate" between the categorical language of the first section and the groupoid language of this section — in either direction. Making these translations will be a focus of the first few chapters. For now, when we find more than one groupoid arising from a given problem, we can investigate whether they are related by morphisms satisfying properties (i) and (ii).

## 3. Triangles

M. Artin has suggested that a good way to get a feeling for stacks is to work out what the moduli space of ordinary triangles should be.

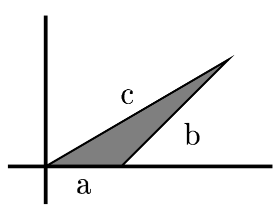
We first consider triangles up to isometry. The classical moduli space would be simply the set $T$ of such triangles, up to isometry, suitably topologized. In categorical language, we take $\mathcal{S}$ to be the category of topological spaces, and define a category $\mathcal{T}$ whose objects are families of triangles. By a family of triangles we mean a continuous and proper map $X \to S$ , making $X$ a fiber bundle over $S$, with a continuously varying metric on the fibers, such that each fiber $X_s$ is (isometric to) a triangle. A morphism from $X' \to S'$ to $X \to S$ is given by a pair of (continuous) maps $X' \to X$ and $S' \to S$ such that the diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

commutes, and so that the induced maps on the fibers are isometries. The functor from $\mathcal{T}$ to $\mathcal{S}$ is the evident one, as in the examples of the first section.

If we consider *ordered* triangles, ordering the sides (or, equivalently, their opposite vertices), then the problem is easy. Here the objects of the corresponding category $\widetilde{\mathcal{T}}$ are fibrations $X \to S$ as before, together with three continuous sections $\alpha$, $\beta$, and $\gamma$ from $S$ to $X$ that pick out the three vertices of each fiber; the morphisms are required to be compatible with these sections. Here the moduli space is the space $\widetilde{T}$ of triples of positive numbers $(a, b, c)$ such that the sum of any two is larger than the third. This $\widetilde{T}$ is the interior of the cone in $\mathbb{R}^3$ spanned by the vectors $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$. The isosceles triangles form three planar subcones; their intersection is the central ray through $(1, 1, 1)$, which parametrizes equilateral triangles.

There is a universal family $\widetilde{Y} \subset \mathbb{R}^2 \times \widetilde{T}$, with its projection $\widetilde{Y} \to \widetilde{T}$, and with the fiber over $(a, b, c)$ in $\widetilde{T}$ being the triangle



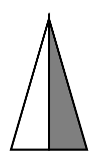

The point is that any triangle with labeled edges of lengths $a$, $b$, and $c$ is *canonically* isometric to this one: given any family $X \to S$ (with three vertex sections), there is a unique map from $S$ to $\widetilde{T}$, and a unique isomorphism of $X$ with the pullback of this universal family. This is what it means, in classical language, to say that $\widetilde{T}$ is a *fine moduli space*. In the new language, this will say that the stack $\widetilde{\mathcal{T}}$ is isomorphic to the stack $\underline{\widetilde{T}}$ of the space $\widetilde{T}$.

If we want a moduli space for unordered triangles, however, the situation is more complicated. The symmetric group $\mathfrak{S}_3$ acts on $\widetilde{T}$ (on the right) by permuting the coordinates, and the quotient space $T = \widetilde{T}/\mathfrak{S}_3$ is the obvious candidate for a moduli space of triangles. Its points, at least, do correspond to triangles up to isometry. The group $\mathfrak{S}_3$ also acts on $\widetilde{Y}$, compatibly with its projection to $\widetilde{T}$.

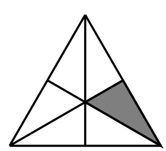
We can therefore construct $Y = \widetilde{Y}/\mathfrak{S}_3$, with an induced map $Y \to T$, which is a natural candidate to be a universal family.

Any family of triangles $X \to S$ will determine a map from $S$ to $T$, but the family may not be isomorphic to the pullback of the "universal" family $Y \to T$. (This moduli space $T$ is a *coarse*, but not a fine, moduli space, for $\mathcal{T}$.) For example, consider a family $X \to S$ of equilateral triangles over a circle $S$ that rotates the triangle by 120° in one revolution around the circle. This is not a constant family, although the corresponding map from $S$ to $T$ is constant.

As this example indicates, the problem with $T$ comes from those triangles that have nontrivial automorphisms. Over a point $t$ of $T$ corresponding to a triangle with sides of different lengths there are six points of $\widetilde{T}$, and elements of $\mathfrak{S}_3$ map the corresponding six triangles in $\widetilde{Y}$ isomorphically to each other; the fiber of $Y$ over $t$ can therefore be identified with the corresponding triangle. For an isosceles triangle, say with sides of lengths 1, 2, and 2, corresponding to a point $t$ in $T$, there are three points $(1,2,2)$, $(2,1,2)$, and $(2,2,1)$ in $\widetilde{T}$ lying over $t$. This time the action of the group includes flips over the altitude, and so the fiber of $Y$ over $t$ can be identified the quotient of the triangle by this flip:



For an equilateral triangle, there is only one point of $\widetilde{T}$ over the point $t$ in $T$, and the fiber of $Y$ over $t$ is the quotient of the triangle by this action of $\mathfrak{S}_3$:



Exercise 1.3. Let $Y^\circ \to T^\circ$ be the restriction of $Y \to T$ to the locus $T^\circ$ of triangles with sides of distinct lengths. Show that $Y^\circ \to T^\circ$ is a fiber bundle and gives a universal family: $T^\circ$ is a fine moduli space for such triangles.

Given any family $X \to S$ of (unordered) triangles, let $\widetilde{S}$ be the set of pairs

$$(s, \text{ordering of edges of } X_s).$$

Then $\widetilde{S} \to S$ is a 6-sheeted covering space, in fact, a principal bundle (torsor) for the group $\mathfrak{S}_3$. If $\widetilde{X} \to \widetilde{S}$ is the pullback of the given family $X \to S$ by the covering map

$\widetilde{S} \to S$, we have a commutative diagram

$$\begin{array}{ccc} \widetilde{S} & \longrightarrow & \widetilde{T} \\ \downarrow & & \downarrow \\ S & \longrightarrow & T \end{array}$$

where the map $\widetilde{S} \to \widetilde{T}$ commutes with the action of $\mathfrak{S}_3$. This is exactly the data for an object of the stack $[\widetilde{T}/\mathfrak{S}_3]$ described in the first section: the stack $\mathcal{T}$ is isomorphic to the quotient stack $[\widetilde{T}/\mathfrak{S}_3]$. (The reader may verify that the functor from $\mathcal{T}$ to $[\widetilde{T}/\mathfrak{S}_3]$ is an equivalence of categories.) The transformation groupoid $\widetilde{T} \times \mathfrak{S}_3 \rightrightarrows \widetilde{T}$ will be an atlas for this stack.

As in this example, it frequently happens that a moduli space can be constructed as a quotient $U/G$ of a space $U$ by the action of a group $G$. This crude quotient space cannot capture the geometry of the moduli problem near points $u$ of $U$ where the stabilizer $G_u = \{g \in G \mid g \cdot u = u\}$ is not trivial. The stack is designed to remember some part of the group action. The group action is not part of the information carried by the stack, however. Indeed, if it were, we would just be studying equivariant spaces.

Here is quite a different atlas for the same stack. By a *plane triangle* we mean a triangle embedded in $\mathbb{R}^2$. Let $G$ be the Lie group of isometries of $\mathbb{R}^2$, which is the 3-dimensional group generated by rotations, reflections, and translations. Let $V$ be the space of (unordered) plane triangles, which is a 6-dimensional manifold.[3] We have a universal family $Z \subset \mathbb{R}^2 \times V$ of plane triangles over $V$. Note that $G$ acts on the left on $V$, and on $\mathbb{R}^2 \times V$, preserving $Z$.

We claim that the stack $\mathcal{T}$ is isomorphic to the quotient stack $[G\backslash V]$. Indeed, if $X \to S$ is an object of $\mathcal{T}$, there is a principal (left) $G$-bundle $E \to S$, whose fiber over $s$ is the space of all isometric embeddings of the fiber $X_s$ into $\mathbb{R}^2$. (Note that this $G$-torsor is trivial over any open set of $S$ on which the $\mathfrak{S}_3$-covering $\widetilde{S} \to S$ is trivial.) We have a $G$-equivariant map from $E$ to $V$, since any point of $E$ determines a plane triangle. This gives a functor from $\mathcal{T}$ to $[G\backslash V]$, which is an equivalence of categories. Summarizing, we have isomorphisms of stacks:

$$[G\backslash V] \cong \mathcal{T} \cong [\widetilde{T}/\mathfrak{S}_3].$$

Note that the two corresponding atlases even have different dimensions. However, $\dim V - \dim G = 6 - 3$ and $\dim \widetilde{T} - \dim \mathfrak{S}_3 = 3 - 0$ are equal; this stack $\mathcal{T}$ will be 3-dimensional.

We can also see a direct relation between the groupoid $G \times V \rightrightarrows V$ and the category $\mathcal{T}$. Any family of triangles is locally planar: if $X \to S$ is a family of triangles, we can choose an open covering $\{S_\alpha\}$ of $S$, with maps $\varphi_\alpha \colon S_\alpha \to V$ and an isomorphism of $X|_{S_\alpha}$ with the pullback of $Z \to V$. On $S_\alpha \cap S_\beta$ there are unique maps $\Phi_{\alpha\beta} \colon S_\alpha \cap S_\beta \to G$ such that $\varphi_\beta(s) = \Phi_{\beta\alpha}(s) \cdot \varphi_\alpha(s)$. This gives a map $\Phi \colon \coprod S_\alpha \cap S_\beta \to G \times V$, taking $s$ in $S_\alpha \cap S_\beta$ to $\Phi_{\beta\alpha}(s) \times \varphi_\alpha(s)$.

[3] This can be constructed as a quotient of the set $\widetilde{V}$ of noncollinear triples in $(\mathbb{R}^2)^3$ by the action of $\mathfrak{S}_3$. That $V$ is a manifold follows from the general fact that this action is free.

EXERCISE 1.4. Show that $\{\phi_\alpha\}$ and $\{\Phi_{\alpha\beta}\}$ determine a morphism from the groupoid $\coprod S_\alpha \cap S_\beta \rightrightarrows \coprod S_\alpha$ to the groupoid $G \times V \rightrightarrows V$. The first is an atlas for $S$, the second an atlas for $[G\backslash V]$.

The following exercise shows how the two atlases for $\mathcal{T}$ are related.

EXERCISE 1.5. The set $\widetilde{V}$ of noncollinear triples in $(\mathbb{R}^2)^3$ has a right action of $\mathfrak{S}_3$ compatible with the left action of $G$. Construct morphisms of groupoids from the groupoid $G\times\widetilde{V}\times\mathfrak{S}_3 \rightrightarrows \widetilde{V}$ to the groupoid $G\times V \rightrightarrows V$ and to the groupoid $\widetilde{T}\times\mathfrak{S}_3 \rightrightarrows \widetilde{T}$, and show that they satisfy conditions (i) and (ii) from the end of §1.2.

EXERCISE 1.6. How do the results of this section change of one replaces isometry (congruence) of triangles by similarity?

## 4. Conics

We want to classify *conics*; for us a conic will be a curve which is isomorphic to the plane curve defined by a homogeneous polynomial of degree two in $\mathbb{P}^2$. Here we take $\mathcal{S}$ to be schemes over $\mathbb{C}$.

There are just three isomorphism classes of plane conics. Let $x, y, z$ be the homogeneous coordinates on $\mathbb{P}^2$, and identify a plane conic with the homogeneous polynomial that defines it (identifying two polynomials if one is a nonzero multiple of the other). The isomorphism classes are

(1) $N$: nonsingular conics, e.g. $x^2 + y^2 + z^2$,
(2) $L$: pairs of two different lines, e.g. $xy$,
(3) $D$: double lines, e.g. $x^2$.

Therefore, in some sense the moduli space of plane conics is just a set $\{N, L, D\}$ of three points.

If $M$ were a fine moduli space for conics, then the morphisms from a scheme $S$ to $M$ would be in one-to-one correspondence with the families of conics over $S$. If $M$ were even a coarse moduli space, any family of conics over $S$ would determine a morphism from $S$ to $M$.

We can first see that if $\{N, L, D\}$ is such a moduli space, then it cannot carry the discrete topology. In the one parameter family defined by $xy + tz^2$, for $t \in \mathbb{C}$, the conic $C_t$ is smooth for $t \neq 0$ and a pair of two different lines for $t = 0$. The corresponding map from $\mathbb{C}$ to $\{N, L, D\}$ sends $\mathbb{C} \smallsetminus 0$ to $N$ and $0$ to $L$; this shows that $L$ must be in the closure of $N$. Similarly the family $x^2 + ty^2$, $t \in \mathbb{C}$, shows that $D$ is in the closure of $L$. So we'd want the closed subsets of $\{N, L, D\}$ to be $\emptyset$, $\{D\}$, $\{D, L\}$, and $\{D, L, N\}$. This cannot be a fine moduli space, and it does not carry any information about the automorphisms of the conics; it is not a scheme, so it cannot even be a course solution in algebraic geometry. This illustrates the principle that the geometric points of a stack may tell us very little about it.

There is also a more concrete description, familiar in algebraic geometry. One identifies the space of plane conics with the projective space $\mathbb{P}^5$ of homogeneous polynomials $ax^2 + bxy + cy^2 + dxz + eyz + fz^2$ of degree two in $x, y, z$ modulo multiplication by nonzero scalars; so a plane conic defined by this polynomial is identified with the point $[a : b : c : d : e : f]$ in $\mathbb{P}^5$. The equation $ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$

defines a universal family $Y \subset \mathbb{P}^2 \times \mathbb{P}^5$ over $\mathbb{P}^5$. The group $G = PGL(3)$ of projective linear transformations of $\mathbb{P}^2$ acts on the space of conics: an element of $G$ defines an isomorphism $g\colon \mathbb{P}^2 \to \mathbb{P}^2$. For a conic $Z$ in $\mathbb{P}^2$ the image $g(Z)$ is another conic, and the restriction $g|_Z\colon Z \to g(Z)$ is an isomorphism. Furthermore it is easy to see that two plane conics $Z$, $W$ are isomorphic if and only if $W = g(Z)$ for a suitable $g \in G$, and that in this case the isomorphisms from $Z$ to $W$ are precisely the restrictions $h|_Z$ of those $h \in G$ such that $h(Z) = W$.[4] From this point of view, the moduli space of conics should be a quotient of $\mathbb{P}^5$ by the group $G$, and we may expect the moduli stack to be the quotient stack $[\mathbb{P}^5/G]$.

Here is a categorical description for the stack of planar conics. A family of conics is a projective, flat, morphism $\pi\colon C \to S$, such that each geometric fiber is isomorphic to one of the three plane conics. Such a family comes with[5] a $\mathbb{P}^2$-bundle $P \to S$, with $C$ embedded as a closed subscheme of $P$; locally, over an affine covering $\{S_\alpha\}$ of $S$, there are isomorphisms $P|_{S_\alpha} \cong \mathbb{P}^2 \times S_\alpha$ of $\mathbb{P}^2$-bundles, taking $C|_{S_\alpha}$ to the zeros of a degree 2 homogeneous polynomial which does not vanish identically at any point of $S_\alpha$. If $E \to S$ is the bundle of local isomorphisms of $P$ with $\mathbb{P}^2$, then $E$ is a principal $G$-bundle over $S$, and we have a $G$-equivariant morphism from $E$ to $\mathbb{P}^5$ that takes a point $s$ to the image of $C_s \subset P_s$, which is a conic in $\mathbb{P}^2$, i.e., a point in $\mathbb{P}^5$. This pair $(E \to S,\ E \to \mathbb{P}^5)$ is an object of the category $[\mathbb{P}^5/G]$, and indicates why the stack of planar conics should be isomorphic to the quotient stack $[\mathbb{P}^5/G]$.

An important part of a moduli problem is to describe the automorphism groups of its objects. When the solution is a quotient by a group action, this is the same as describing the stabilizer of a representative point. For conics, we have the three cases:

(1) $N$: $x^2+y^2+z^2$; the stabilizer consists of the complex orthogonal $3\times 3$ matrices (i.e, those $A$ such that ${}^tA \cdot A = I$, up to scalars . This group has dimension 3.

(2) $L$: $xy = 0$; the stabilizer consists of all invertible $3 \times 3$ matrices $A$ modulo scalars, where $A$ is of the form

$$\begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ 0 & 0 & * \end{pmatrix}.$$

It has dimension 4.

(3) $D$: $x^2 = 0$; the stabilizer is the set of all matrices of the form

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

Its dimension is 6.

---

[4]The group $G = PGL(n+1)$ acts on the left on $\mathbb{P}^n$, so it acts on the right on the polynomials $\Gamma(\mathbb{P}^n, \mathcal{O}(m))$ of degree $m$ by the formula $(F \cdot g)(x) = F(g \cdot x)$.

[5]In fact, $P$ may be taken to be the projective bundle $\mathbb{P}(\pi_*(\omega_{C/S}^\vee)$ of lines in the rank 3 vector bundle $\pi_*(\omega_{C/S}^\vee)$, where $\omega_{C/S}$ is the relative dualizing sheaf. A reader to whom this is unfamiliar can take this added structure of an embedding in a $\mathbb{P}^2$-bundle as part of the definition.

In general, if a smooth algebraic group $G$ of dimension $k$ acts freely on a smooth variety of dimension $d$, then the quotient (when it exists) will be smooth of dimension equal to $d - k$. This is not true in the naive world if the action is not free, but it will remain true in the stack world. A smooth orbit for the action will correspond to a "point" in the quotient and the dimension of this point will be equal to the dimension of the orbit minus $k$.

In the case of conics, $G = PGL(3)$ has dimension 8, and the action of $G$ on $\mathbb{P}^5$ has three orbits corresponding to the isomorphism types $N, L, D$ of conics. The corresponding orbits are smooth of dimensions 5, 4 and 2 respectively. Therefore the quotient consists of three points: an open one, $N$, of dimension $5 - 8 = -3$; in its closure another point $L$ of dimension $4 - 8 = -4$; and in its closure the point $D$ of dimension $2 - 8 = -6$. Note that these dimensions are precisely the negatives of the dimensions of the automorphism groups of conics in $N$, $L$ and $D$ respectively. (For an algebraic group $H$, the dimension of the stack $BH = [\bullet/H]$ will be minus the dimension of $H$.)

As with triangles, the atlas we have given is only one of many. For example, one could take $U$ to be the conics passing through the point $[0 : 0 : 1]$. This is a hyperplane in $\mathbb{P}^5$, defined by the vanishing of the coefficient of $z^2$. Take $R$ to be the subset of $U \times G$ consisting of those pairs $(u, g)$ such that $u \cdot g$ is also in $U$. Then there is a natural groupoid structure $s$, $t$, $e$, $m$, $i$ on $U$ and $R$ so that the inclusion of $U$ in $\mathbb{P}^5$ and the inclusion of $R$ in $\mathbb{P}^5 \times G$ determines a morphism of groupoids from $R \rightrightarrows U$ to $\mathbb{P}^5 \times G \rightrightarrows \mathbb{P}^5$; this morphism satisfies the two conditions at the end of §1.2. Note that this groupoid is not of the form $U \times H \rightrightarrows U$, for any action of a group $H$ on $U$.

## 5. Elliptic curves

Elliptic curves have been a fruitful area for the development of moduli problems, as well as stacks (e.g., [**71**], [**21**]). We will devote Chapter 13 to elliptic curves, including the cases of arbitrary characteristic and over $\mathbb{Z}$. Here we sketch a few of the ideas, working in the category $\mathcal{S}$ of schemes over $\mathbb{C}$.

It is known classically that an elliptic curve $E$ over $\mathbb{C}$ is classified up to isomorphism by a value $j \in \mathbb{C}$ known as the *$j$-invariant*, and all complex numbers occur. The (coarse) moduli space for isomorphism classes of elliptic curves should therefore be $\mathbb{C}$ (the complex plane, or, to an algebraic geometer, the affine line $\mathbb{A}^1$). However, the $j$-line is not a fine moduli space, as we will soon see; and, in fact, no fine moduli space exists.

A family of elliptic curves is a smooth and proper morphism $C \to S$, with all geometric fibers $C_s$ connected curves of genus 1, with a section $S \to C$. We often abbreviate this data to $C \to S$, or sometimes even to $C$. A morphism from $C' \to S'$ to $C \to S$ is pair of morphisms $C' \to C$ and $S' \to S$ such that the diagrams

$$\begin{array}{ccc} C' & \longrightarrow & C \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array} \qquad\qquad \begin{array}{ccc} S' & \longrightarrow & S \\ \downarrow & & \downarrow \\ C' & \longrightarrow & C \end{array}$$

commute, and the first (and therefore the second) is cartesian. This determines the category $\mathcal{M}_{1,1}$, with the evident functor from $\mathcal{M}_{1,1}$ to $\mathcal{S}$.

The section determines an origin in each fiber, which then gets the structure of an abelian variety; the section is called the zero section, or the identity section.[6] Note that every elliptic curve comes with an involution, written $p \mapsto -p$, that takes a point to its inverse with respect to this group structure. For instance, if $f(x)$ is a cubic polynomial over the complex numbers with 3 distinct roots, then $y^2 = f(x)$ is the equation of (an affine model of) an elliptic curve $E$. When $S = \operatorname{Spec}\mathbb{C}$ and $C$ is the elliptic curve $E$, the identity section is the point at infinity, and the involution sends $(x, y)$ to $(x, -y)$.

One reason that $\mathbb{A}^1$ is not a fine moduli space is because there are non-trivial families whose fibers are all isomorphic – so-called *isotrivial* families. The corresponding map from $S$ to a moduli space would be constant, and, if the moduli space were fine, the family would have to be trivial.

EXERCISE 1.7. Let $E$ be the elliptic curve given by $y^2 = f(x)$, as above. Let $S = \mathbb{A}^1 \setminus \{0\}$, with coordinate $\lambda$, and let $C \to S$ be the family of elliptic curves given by $\lambda y^2 = f(x)$. (1) Every fiber of this family is isomorphic to E. (2) This family has only finitely many sections, and, in particular, is not isomorphic to the trivial family $S \times E \to S$.

Another reason that $\mathbb{A}^1$ cannot be a fine moduli space is that there are natural line bundles that one arise naturally on a scheme $S$ whenever one is given a family of elliptic curves on $S$. One such line bundle assigns to a family $C \to S$ the normal bundle to the section $S \to C$. This assignment is natural in that sense that a morphism from $C' \to S'$ to $C \to S$ determines an isomorphism of the line bundle on $S'$ with the pullback of the corresponding line bundle on $S$. (Such data, with an appropriate compatibility condition, is what is meant by a line bundle on the stack $\mathcal{M}_{1,1}$.) One can (and we will) see that such line bundles are not always trivial. But there are no nontrivial (algebraic or analytic) line bundles on $\mathbb{A}^1$, so these line bundles cannot be pulled back via a morphism to $\mathbb{A}^1$.

One can study such line bundles without the formal language of stacks, and this is what Mumford did in [**71**]. He showed that there are exactly 12 such bundles (up to isomorphism), all tensor powers of the one just discussed. In stack language, this will say that $\operatorname{Pic}(\mathcal{M}_{1,1})$ is $\mathbb{Z}/12\mathbb{Z}$.

In this seminal paper, Mumford introduced the notion of a "modular family". This is a collection $\{\pi_\alpha\colon C_\alpha \to S_\alpha\}$ of families of elliptic curves, with the following property: Each $S_\alpha$ must be a smooth curve, and, moreover, any first order deformation of the fiber of $\pi_\alpha$ at $s \in S_\alpha$ must be captured by some tangent to $S_\alpha$ at $s$. The idea is that these $\{S_\alpha\}$ should be étale over the ideal moduli space (which cannot exist). This can

[6] In fact, the family gets the structure of a group scheme over $S$ (see [**48**], §2)

be expressed by the assertion that, for any diagram

$$\begin{array}{ccc} \mathrm{Spec}(\Lambda/I) & \longrightarrow & S_\alpha \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(\Lambda) & \longrightarrow & mathcal M_{1,1} \end{array}$$

with $\Lambda$ an Artin $\mathbb{C}$-algebra, $I$ an ideal in $\Lambda$ such that $I^2 = 0$, the dotted arrow can be filled in uniquely to make the diagram commute. Precisely, this says that for any family $\pi$ of elliptic curves over $\Lambda$, and any morphism $\overline{f}\colon \mathrm{Spec}(\Lambda/I) \to S_\alpha$, and an isomorphism $\overline{\vartheta}$ of $\pi \otimes_\Lambda \Lambda/I$ with $\overline{f}^*(\pi_\alpha)$, there is a unique isomorphism $f\colon \mathrm{Spec}(\Lambda) \to S_\alpha$ lifting $\overline{f}$ and a unique $\vartheta$ of $\pi$ with $f^*(\pi_\alpha)$ lifting $\overline{\vartheta}$[7]

A modular family $\{\pi_\alpha\colon C_\alpha \to S_\alpha\}$ can be called a *covering* if every elliptic curve is isomorphic to some fiber of some $\pi_\alpha$. (This makes $\{S_\alpha\}$ an étale covering of the nonexistent moduli space.) space)

We will see how a covering modular family determines an algebraic groupoid, which in fact is an atlas for the moduli stack.

To construct modular families, we need a few facts from the theory of elliptic curves, as found say in [**85**] when the base is a field, supplemented by [**19**] or [**48**] for families. Any elliptic curve can be embedded in the projective plane, with its chosen origin taken to the point $[0 : 1 : 0]$, and with an equation $y^2z = x^3 + Axz^2 + Bz^3$, with $A$ and $B$ complex numbers such that the form on the right vanishes at three distinct points in the projective line. We write this in affine coordinates:

$$y^2 = x^3 + Ax + B, \qquad (4A^3 + 27B^2 \neq 0).$$

This is a family over $E \subset \mathbb{P}^2 \times W$ over $W = \{(A, B) \in \mathbb{C}^2 \mid 4A^3 + 27B^2 \neq 0\}$, called a *Weierstrass family.*[8] The $j$-invariant is given by[9]

$$j = 1728 \cdot \frac{4A^3}{4A^3 + 27B^2}.$$

All $j$-invariants (elliptic curves) occur in this family, but it is 2-dimensional. We will look at some 1-dimensional families by restricting this to various lines, first a diagonal line, and then a horizontal and a vertical line.

The family

$$C_0 : \qquad y^2 = x^3 + \frac{27}{4} \cdot \frac{j}{1728 - j}(x + 1)$$

over $S_0 = \mathbb{A}^1 \smallsetminus \{0, 1728\}$ has the virtue that the $j$-invariant the fiber over $j$ is $j$. However, this family cannot be extended smoothly across the two missing points. In

[7] This condition makes $S_\alpha$ what is called a universal deformation space at each of its points. Note that the first order deformations of an elliptic curve $E$ are parametrized by $H^1(E, T_E) = H^1(E, \mathcal{O}_E)$, which is 1-dimensional. That each $S_\alpha$ smooth and 1-dimensional follows, in fact, from the lifting property described after **??**.

[8] In fact, any family $C \to S$ of elliptic curves is, locally in the Zariski topology, isomorphic to the pullback of $E \to W$ by a morphism from $S$ to $W$.

[9] In [**71**] Mumford replaces $j$ by $1728 - j$.

fact, any modular family that contains a curve with $j$-invariant 0 or 1728 must have curves with other $j$-invariants that appear more than once. Here we see something "stacky" about this moduli problem, since $j = 0$ and $j = 1728$ are precisely the curves with additional automorphisms besides the identity and the involution $p \mapsto -p$.

Consider the family $C_1 \to S_1$ with

$$C_1 : \qquad y^2 = x^3 + Ax + 1$$

and $A \in S_1 = \{A \in \mathbb{A}^1 \mid 4A^3 + 27 \neq 0\}$. This attains every $j$-invariant except $j = 1728$. The family $C_2 \to S_2$ with

$$C_2 : \qquad y^2 = x^3 + x + B$$

with $B \in S_2 = \{B \in \mathbb{A}^1 \mid 4 + 27B^2 \neq 0\}$ attains every $j$-invariant but $j = 0$.

Exercise 1.8. These two families satisfy the conditions of **??**.

Together, these two families form a covering modular family.

Exercise 1.9. Show that the morphism $j$ from $S_1 \amalg S_2$ to the affine line is unramified except over 0 and 1728, and show that the ramification index is 3 over $j = 0$ and 2 over $j = 1728$.

To make an atlas, we want to glue $S_1$ and $S_2$, to keep track of where an elliptic curve appears in both families. In the stack world, we don't just take an equivalence relation on $S_1 \amalg S_2$; rather, we keep track of automorphisms. That is, for $\alpha$ and $\beta$ in $\{1, 2\}$, we set

$$R_{\alpha,\beta} = \{(u, v, \phi) \mid u \in S_\alpha, v \in S_\beta, \text{ and } \phi\colon (C_\alpha)_u \xrightarrow{\cong} (C_\beta)_v\}.$$

This comes with projections $s\colon R_{\alpha,\beta} \to S_\alpha$, taking $(u, v, \phi)$ to $u$, and $t\colon R_{\alpha,\beta} \to S_\alpha$, taking $(u, v, \phi)$ to $v$. Define

$$U = S_1 \amalg S_2$$

and take $R$ to be the disjoint union of these four $R_{\alpha,\beta}$:

$$R = R_{1,1} \amalg R_{1,2} \amalg R_{2,1} \amalg R_{2,2}.$$

Then we have maps $s$ and $t$ from $R$ to $U$. The multiplication $m$ comes by composing the isomorphisms, taking $(u, v, \phi) \times (v, w, \psi)$ to $(u, w, \psi \circ \phi)$. The identity $e$ takes $u \in S_\alpha$ to $(u, u, \mathrm{id})$, where id is the identity map on $(C_\alpha)_u$; and the inverse $i$ takes $(u, v, \phi)$ to $(v, u, \phi^{-1})$. It is a straightforward exercise to verify that this forms a groupoid $R \rightrightarrows U$. This will be an atlas for the stack $\mathcal{M}_{1,1}$.

If two elliptic curves are given in Weierstrass form, $y^2 = x^3 + Ax + B$ and $y^2 = x^3 + A'x + B'$, it is a general fact that any isomorphism between them must be of the form $(x, y) \mapsto (\lambda x, \mu y)$ for some $\lambda$, $\mu \in \mathbb{C}^*$ (see [**85**], §III.3, [**19**], §1). So, for instance, we can express $R_{1,1}$ as the variety consisting of all $\{(A, A', \lambda, \mu)\}$ such that

$$\mu^2 x^3 + \mu^2 Ax + \mu^2 = \lambda^3 x^3 + \lambda A' x + 1.$$

In particular, $\mu^2 = 1$ and $\lambda^3 = 1$; setting $\gamma = \mu/\lambda$ we have $A' = \gamma^4 A$ and $\gamma$ can be any $6^{\text{th}}$ root of unity. Let $\phi_{[\gamma]}$ denote the map $(x, y) \mapsto (\gamma^2 x, \gamma^3 y)$. Then

$$R_{1,1} \cong S_1 \times \mu_6$$

by associating $(A, \gamma^4 A, \phi_{[\gamma]})$ in $R_{1,1}$ to $(A, \gamma)$ in $S_1 \times \mu_6$.

EXERCISE 1.10. Deduce, in a similar fashion, that $R_{2,2}$ is isomorphic to $S_2 \times \mu_4$ and that $R_{1,2}$ and $R_{2,1}$ can each be identified with the complement of 13 points in the affine line. (In fact the isomorphisms of curves can all be expressed conveniently using the $\phi_{[\gamma]}$ notation.)

We want to see how this groupoid $R \rightrightarrows U$ can tell us about moduli of elliptic curves, i.e., about the category $\mathcal{M}_{1,1}$. We have a family $C \to U$, with $C = C_1 \amalg C_2$, and this contains every elliptic curve at least once. For any $S$ and any map $\varphi\colon S \to U$, we can pull back this family $C \to U$ to get a family on $S$, namely $C \times_U S \to S$. However, this fails two basic criteria to be a universal family:

(1) Two different maps $\varphi_1\colon S \to U$ and $\varphi_2\colon S \to U$ may determine isomorphic families on $S$.
(2) Some families over $S$ may not be pullbacks from any morphisms from $S$ to $U$.

As far as (1) is concerned, an isomorphism from the first pullback to the second determines (and is determined by) a morphism $\psi\colon S \to R$ that takes a point $s$ to the given isomorphism from $C_{\varphi_1(s)}$ to $C_{\varphi_2(s)}$. In short, we have

$$\psi\colon S \to R \quad \text{with} \quad s \circ \psi = \varphi_1 \quad \text{and} \quad t \circ \psi = \varphi_2.$$

An extreme example of this occurs with $S = R$, $\varphi_1 = s$, $\varphi_2 = t$, in which case $\psi$ is the identity.

The family $C_0 \to S_0 = S$ that we started with is an example of the failure of (2): it is not the pullback from any map from $S$ to $U$. However, it is *locally* a pullback: near any $j$ in $S$, there is a disk $\Delta$ containing $j$, and a morphism $\Delta \to U$ so that the restriction of the family to $\Delta$ is isomorphic to the pullback of the family $C \to U$. This works in the analytic category, but not in the algebraic, if one uses the Zariski topology. Indeed, the only nonempty Zariski open sets in $S$ are the complements of finite sets. But one can find a variety $T$, with a surjective morphism $\rho\colon T \to S$, which is locally an analytic isomorphism — this makes it *étale* — together with a morphism $\varphi\colon T \to U$, with an isomorphism $\vartheta$ from the pullback of $C_0 \to S$ to $T$ via $\rho$ with the pullback of $C \to U$ via $\varphi$.

EXERCISE 1.11. Show that $T = \{a \in \mathbb{A}^1 C \mid a \neq 0, 4a^6 + 27 \neq 0\}$ is such a variety (with family $y^2 = x^3 + a^2 x + 1$), and with the map $\rho\colon T \to S$ given by $a \mapsto 1728 \cdot 4a^6/(4a^6 + 27)$, and $\varphi\colon T \to S_1 \subset U$ given by $a \mapsto A(a) = a^2$.

For such a "covering" $\rho\colon T \to S$ (or $T$ a disjoint union of disks in the analytic case), we have a groupoid $T \times_S T \rightrightarrows T$. For any point $(t, t')$ in $T \times_S T$, with $s$ their common image in $S$, we have isomorphisms $C_t \cong (C_0)_s \cong C_{t'}$.

EXERCISE 1.12. Show that these isomorphisms are given by a global isomorphism of $(\varphi \circ p_1)^*(C)$ with $(\varphi \circ p_2)^*(C)$ on $T \times_S T$. This defines a morphism $\Phi$ from $T \times_S T$ to $R$ with $s \circ \Phi = \varphi \circ p_1$ and $t \circ \Phi = \varphi \circ p_2$. Show that $(\varphi, \Phi)$ determines a morphism from the groupoid $T \times_S T \rightrightarrows T$ to the groupoid $R \rightrightarrows U$.

Note that $T \times_S T \rightrightarrows T$ is an atlas for $S$, and $R \rightrightarrows U$ is supposed to be an atlas for $\mathcal{M}_{1,1}$, so the groupoid morphism of the exercise can be regarded as a geometric realization of the morphism from (the stack corresponding to) $S$ to the stack $\mathcal{M}_{1,1}$.

This picture can be reversed. Given an étale surjective map $\rho\colon T \to S$, and a morphism $(\varphi, \Phi)$ from $T \times_S T \rightrightarrows T$ to the groupoid $R \rightrightarrows U$, one gets a family $\varphi^*(C)$ of elliptic curves on $T$ and an isomorphism $p_1^*(\varphi^* C) \xrightarrow{\sim} p_2^*(\varphi^* C)$ on $T \times_S T$, satisfying a compatibility identity on $T \times_S T \times_S T$. It is the theory of *descent* that implies that such a family is the pullback of a family on $S$.

This example contains another fundamental insight of Grothendieck: to replace Zariski open coverings $\{S_\alpha\}$ of a variety or scheme $S$ not just by $\coprod S_\alpha \to S$, but by an arbitrary collections of étale morphisms $S_\alpha \to S$ whose images cover $S$. When the base category is a category of schemes, the topology we will usually use — the *étale topology* — has these étale maps as its basic open sets.

The theory of descent is again called on to justify the assertion that $\mathrm{Pic}(\mathcal{M}_{1,1})$ can be identified with the group of line bundles $L$ on $U$ equipped with isomorphisms $s^*L \xrightarrow{\sim} t^*L$ on $R$, satisfying a natural compatibility condition on $R\ {}_t\!\times_s R$, up to compatible isomorphism of line bundles. The latter group is described by a finite amount of data: $U$ has no nontrivial line bundles (since its components are Zariski open subsets of the affine line), so without loss of generality, $L$ may be assumed trivial. Now an isomorphism $s^*L \xrightarrow{\sim} t^*L$ is just an invertible function on $R$. So, $\mathrm{Pic}(\mathcal{M}_{1,1})$ is the quotient of the group of elements of $\mathcal{O}^*(R)$ satisfying the compatibility condition on $R\ {}_t\!\times_s R$ by the subgroup of elements of the form $t^*\chi/s^\times\chi$, with $\chi \in \mathcal{O}^\times(U)$. A tedious but doable calculation yields the isomorphism $\mathrm{Pic}\,\mathcal{M}_{1,1} \cong \mathbb{Z}/12\mathbb{Z}$. This calculation will be carried out, using slightly different atlases, in Chapter 13.

In the analytic category, one has a modular family $E \to \mathbb{H}$ of elliptic curves over the upper half plane $\mathbb{H}$ whose fiber over $\tau$ in $\mathbb{H}$ is the elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$, with $\Lambda_\tau$ the lattice $\mathbb{Z} + \mathbb{Z}\cdot\tau$. An isomorphism from $E_\tau$ to $E_{\tau'}$ is given by multiplication by a unique complex number $\vartheta$ such that $\vartheta \cdot \Lambda_\tau = \Lambda_{\tau'}$. A corresponding atlas is the groupoid $R \rightrightarrows \mathbb{H}$, where $R = \{(\tau, \tau', \vartheta) \in \mathbb{H} \times \mathbb{H} \times \mathbb{C} \mid \vartheta \cdot \Lambda_\tau = \Lambda_{\tau'}\}$. In fact, Mumford uses this analytic modular family in [**71**] to give a calculation of $\mathrm{Pic}(\mathcal{M}_{1,1}) \cong \mathbb{Z}/12\mathbb{Z}$ in the analytic category.

EXERCISE 1.13. Show that this analytic groupoid $R \rightrightarrows \mathbb{H}$ is isomorphic to the transformation groupoid $SL_2(\mathbb{Z}) \ltimes \mathbb{H}$ coming from the standard action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$:

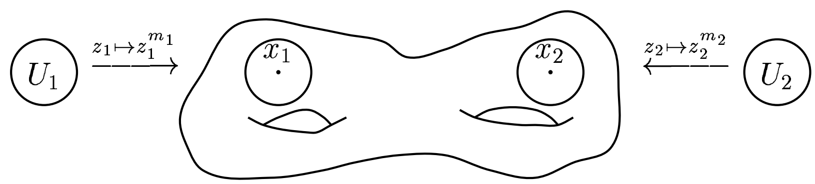
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

If one considers only elliptic curves with $j$-invariant not 0 or 1728, one has a category $\mathcal{M}^\circ$, with a canonical map $j$ from $\mathcal{M}^\circ$ to $\mathbb{A}^1 \smallsetminus \{0, 1728\}$. This is not an isomorphism, because every family has a non-trivial automorphism, given by $p \mapsto -p$ in each fiber. We will see, in fact, that $\mathcal{M}^\circ$ is isomorphic to the product of $\mathbb{A}^1 \smallsetminus \{0, 1728\}$ and the stack $B(\mathbb{Z}/2\mathbb{Z})$.

## 6. Orbifolds

Orbifolds, sometimes called $V$-manifolds, provide another good introduction to some of the notions involved with stacks. In fact, the moduli stack of triangles, or any situation where a finite group acts on a manifold, gives rise to an orbifold. An orbifold is often described as a space that is locally a quotient of a manifold by a finite group, but this description is too crude: to give an orbifold, one must describe these local group actions, at least up to some equivalence. We will see that this extra data amounts to the difference between an ordinary space and a stack. (In fact, the underlying space corresponds to the coarse moduli space of the stack.)

As a simple example, let $X$ be a Riemann surface, and let $x_1, \dots, x_n$ be a finite set of points of $X$, and let $m_1, \dots, m_n$ be positive integers. Take a neighborhood $V_i$ of $x_i$ biholomorphic to a disk, and choose an isomorphism $V_i \cong U_i/G_i$, where $U_i$ is a disk, and $G_i$ is the cyclic group of $m_i^{\text{th}}$ roots of unity, acting by rotation; take all the neighborhoods $V_i$ to be disjoint.




At any other point $x$ of $X$, choose any neighborhood of $x$ biholomorphic to a disk and not containing any of the points $x_i$. These data determine an orbifold structure on the Riemann surface. Although the underlying (coarse) space is the original surface $X$, the orbifold structure is different, at any point $x_i$ with $m_i > 1$. (See [**69**] for more on these Riemann surface orbifolds.)

For an explicit example, let $X = S^2 = \mathbb{C} \cup \{\infty\}$, with one point $p_1 = \infty$, and with $m(p_1) = m$. This is sometimes called the *m-teardrop*.

We turn next to a precise definition of an orbifold, following Haefliger [**43**], §4. (Compare Kawasaki's variation [**49**] of Satake's original [**80**].) We will define a complex analytic orbifold, although similar constructions work in other categories, cf. [**70**].

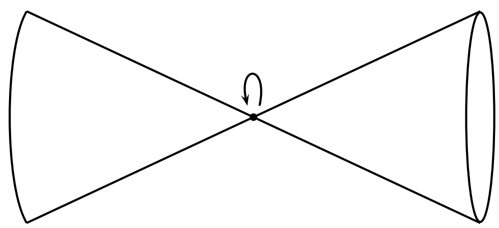
One starts with a topological space $X$. The data to give an orbifold structure to $X$ consists of an open covering $\{V_\alpha\}$ of $X$, together with homeomorphisms $V_\alpha \cong G_\alpha \backslash U_\alpha$, where $U_\alpha$ is a connected complex manifold (usually taken to be an open set in $\mathbb{C}^n$), $G_\alpha$ is a finite group of analytic automorphisms of $U_\alpha$, and $G_\alpha \backslash U_\alpha$ denotes the set of orbits, with the quotient topology inherited from $U_\alpha$. (The action of $G_\alpha$ on $U_\alpha$ is assumed to be effective, i.e., $G_\alpha \subset \operatorname{Aut}(U_\alpha)$.) This data must satisfy the following compatibility condition: if $u \in U_\alpha$, and $u' \in U_\beta$ map to the same point in $X$, there must be neighborhoods $W$ of $u$ in $U_\alpha$, and $W'$ of $u'$ in $U_\beta$, and a complex analytic isomorphism $\varphi \colon W \to W'$ taking $u$ to $u'$ and commuting with the projections to $X$:

$$\begin{array}{ccc} U_\alpha \supset W & \xrightarrow{\varphi} & W' \subset U_\beta \\ \searrow & & \swarrow \\ & V_\alpha \subset X \supset V_\beta & \end{array}$$

Note that these germs are not part of the data for the orbifold, and they need not be unique; rather, their existence is a condition on the data. The same idea definess when two such data are compatible; an **orbifold structure** on $X$ is an equivalence class of data, where compatible data are called equivalent. The space $X$ one gets from this data inherits a complex analytic structure; it will be the "coarse" space for the corresponding stack. If $X$ is connected, the manifolds $U_\alpha$ all have the same dimension, called the *dimension* of the orbifold.

From such data one can construct an analytic groupoid, as follows. Set $U = \coprod U_\alpha$, and set $R$ to be the set of triples $(u, u', \varphi)$, where $u$ and $u'$ are points in $U$ with the same image in $X$, and $\varphi$ is a germ of an isomorphism from a neighborhood of $u$ to a neighborhood of $u'$ over $X$. This $R$ has a unique topology so that the two projections $s$ and $t$ from $R$ to $U$ (taking $(u, u', \varphi)$ to $u$ and $u'$ respectively) are local homeomorphisms; this gives $R$ the structure of a complex manifold. The other maps are easily defined: $e\colon U \to R$ takes $u$ to $(u, u, \mathrm{id})$, $i\colon R \to R$ takes $(u, u', \varphi)$ to $(u', u, \varphi^{-1})$, and $m\colon R\,{}_t\!\times_s R \to R$ takes $(u, u', \varphi) \times (u', u'', \psi)$ to $(u, u'', \psi \circ \varphi)$. When we regard an orbifold as a stack this will be an atlas for it.

The simplest example of orbifold is a finite group quotient. Here $U$ is a manifold, $G$ is a finite group with an effective action on $U$, and $V$ is the quotient space $G \backslash U$. In this case $R$ can be identified with $G \times U$ and we recover the transformation groupoid $G \ltimes U$. For instance, if $U = \mathbb{C}^2$ and $G = \mathbb{Z}/2\mathbb{Z}$, with the action of its generator given by $(x, y) \mapsto (-x, -y)$, then the quotient (analytic) space $V$ is a quadric cone, isomorphic to the locus in $\mathbb{C}^3$ defined by the equation $uv = w^2$. In this case the orbifold quotient can be pictured as follows.



At the vertex of the cone there is a nontrivial orbifold structure (indicated by the arrow); the complement is a manifold.

EXERCISE 1.14. Construct a groupoid for the $m$-teardrop. Take $U = U_1 \coprod U_2$, with $U_1 = \mathbb{C}$ and $U_2 = D$ an open disk mapping to a neighborhood of $\infty$ by $z \mapsto 1/z^m$. Compute $R$, and $s$, $t$, $m$, $e$, and $i$.

Note that the canonical map from $R$ to $U \times U$ is never injective, unless all the maps $U_\alpha \to X$ are local homeomorphisms, in which case $X$ is a manifold with its trivial orbifold structure.

EXERCISE 1.15. For any $u$ in $U$, the automorphism group $\mathrm{Aut}(u) = s^{-1}(u) \cap t^{-1}(u)$ is canonically isomorphic to the isotropy group $(G_\alpha)_u = \{g \in G_\alpha \mid g \cdot u = u\}$, if $u$ is in $U_\alpha$. The canonical morphism $R \to U \times U$ is injective if and only if all the isotropy groups are trivial.

For a point $u$ in $U_\alpha$, write $G_u$ for the isotropy group $(G_\alpha)_u = \{g \in G_\alpha \mid g \cdot u = u\}$. Given one germ $\varphi$ from $u$ to $u'$, over a point $x$ in $X$, with $u \in U_\alpha$ and $u' \in U_\beta$, the other possible germs have the form $\varphi \circ g$, where $g$ is in the isotropy group $G_u$; they also have the form $g' \circ \varphi$ for $g'$ in $G_{u'}$. Fixing one such $\varphi$ determines an isomorphism from $G_u$ to $G_{u'}$, sending $g$ to $g'$ when $\varphi \circ g = g' \circ \varphi$. This means that one has can assign an *isotropy group* $G_x$ for each point $x$ in $X$, defined to be $G_u$ for any point $u$ that maps to $x$. This group is determined only up to (inner) isomorphism, since changing $\varphi$ gives another isomorphism of $G_u$ with $G_{u'}$ differing by an inner automorphism. In fact, the map $g \mapsto g_*$, where $g_*$ is the induced endomorphism of the tangent space $T_u U_\alpha \cong \mathbb{C}^n$, gives an embedding $G_u \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ (see [**17**], §4), so we have an embedding of $G_x$ in $\mathrm{GL}_n(\mathbb{C})$, unique up to conjugacy.

It is a general fact (cf. [**80**], p. 475), that any connected orbifold can be written globally as a quotient of a manifold $M$ by a Lie group $G$, in fact, with $G = \mathrm{GL}_n(\mathbb{C})$ in this complex case, with $n$ the dimension of the orbifold. Let us work this out in the language of groupoids. Let $P_\alpha \to U_\alpha$ be the bundle of frames, with fiber over $u \in U_\alpha$ being the set of bases of the tangent space $T_u U_\alpha$. This is a principal right $G$-bundle, with action of $g = (g_{ij})$ on a frame $v = (v_1, \ldots, v_n)$ by $(v \cdot g)_i = \sum_j v_j g_{ji}$. The group $G_\alpha$ acts on the left on $P_\alpha$, by $(\tau \cdot v)_i = \tau_*(v_i)$. This action is free, and commutes with the action of $G$. Therefore the quotient $M_\alpha = G_\alpha \backslash P_\alpha$ is a manifold, and $G$ acts in the right on $M_\alpha$. Let $\rho_\alpha \colon M_\alpha \to V_\alpha$ be the canonical projections. The orbifold data determine gluing maps from $M_\alpha \cap \rho_\alpha^{-1}(V_\alpha \cap V_\beta)$ to $M_\beta \cap \rho_\beta^{-1}(V_\alpha \cap V_\beta)$, taking the class of a frame $v$ to the class of the frame $\varphi_*(v)$, for any choice of local germ $\varphi$. This gluing data commutes with the action of $G$, so we obtain a manifold $M$ with a right action of $G = \mathrm{GL}_n(\mathbb{C})$, and a projection from $M$ to $X$ that is constant on orbits.

To say that the orbifold is the same as the quotient $[M/G]$, we should compare the groupoid $M \times G \rightrightarrows M$ with the groupoid $R \rightrightarrows U$ defining the orbifold structure.

Exercise 1.16. Let $P = \coprod_\alpha P_\alpha$, with canonical projection $\pi \colon P \to U$. Let $Q = \{(v, v', \varphi) \mid v, v' \in P, \varphi \text{ a germ from } \pi(v) \text{ to } \pi(v')\}$. (a) Construct a groupoid $Q \rightrightarrows P$, with $s$ and $t$ taking $(v, v', \varphi)$ to $v$ and $v'$ respectively, and $m((v, v', \varphi), (v', v'', \psi)) = (v, v'', \psi \circ \varphi)$. (b) Construct a morphism from $Q \rightrightarrows P$ to $R \rightrightarrows U$, taking $(v, v', \varphi)$ to $(\pi(v), \pi(v'), \varphi)$, and verify that it satisfies the two conditions of §1.2. (c) Construct a morphism from $Q \rightrightarrows P$ to $M \times G \rightrightarrows M$, taking $(v, v', \varphi)$ to $(v, g)$, where $g$ is determined by the equation $v'_i = \sum_j \varphi_*(v_j) g_{ji}$, and show that this morphism satisfies the same two conditions.

The local charts on an orbifold are used to do analysis (see [**7**] and [**34**]). For example, a differential form is given by a compatible collection of $G_\alpha$-invariant differential forms $\omega_\alpha$ on $U_\alpha$. In terms of the groupoid, this is a differential form $\omega$ on $U$ such that $s^*(\omega) = t^*(\omega)$ on $R$. In fact, groupoids provide a useful setting for much of the study of orbifolds see [**33**].

It should perhaps be pointed out that some authors also use a more restricted notion of orbifold, where the groups $G_\alpha$ are not allowed to include any complex reflections (i.e isomorphisms conjugate to those of the form $(z_1, \ldots, z_n) \mapsto (\zeta z_1, \ldots, z_n)$, where $\zeta$ is a root of unity, cf. [**78**]); in this case the coarse space $X$ actually determines the orbifold. This rules out orbifold structures like the one we gave on a Riemann surface at the beginning of this section, however. We have seen a similar phenomenon for elliptic curves. where the $j$-line is a coarse moduli space, but the stack "remembers" the automorphisms of the elliptic curves.

The definition we have given here works also for differentiable or topological orbifolds, by replacing the word "complex analytic" by "differentiable" or "continuous", cf.

[**70**]. One can give a corresponding definition in algebraic geometry, although here one must use étale neighborhoods to describe a notion of germ of an isomorphism. There are more general notions of orbifolds, cf. [**84**], where it is not required that the action of each $G_\alpha$ on $U_\alpha$ be effective. Both of these notions can be described more easily in the language of stacks.

## 7. Vector bundles

We fix a rank $n$, and consider complex vector bundles $E \to S$ of rank $n$ on a topological space $S$. We can think of this as a continuously varying family of $n$-dimensional vector spaces. This is an object in a category $\mathcal{V}_n$ over the category $\mathcal{S}$ of topological spaces. One can similarly consider $C^\infty$, complex-analytic, or algebraic vector bundles, with $\mathcal{S}$ the category of $C^\infty$-manifolds, complex-analytic spaces, or algebraic varieties. A morphism from $E' \to S'$ to $E \to S$ is again given by a cartesian diagram

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

with induced maps on fibers isomorphisms of vector spaces; equivalently, we have an isomorphism of $E'$ with the pullback of $E$ by the morphism $S' \to S$.

There is no possibility of a universal family $E_U \to U$. If there were, the vector space $\mathbb{C}^n$ over a point would correspond to a point $p \in U$. Any trivial bundle on a space $S$ is a pullback of $\mathbb{C}^n$ on a point, so the corresponding map $S \to U$ would have to map $S$ to $p$. But for any vector bundle $E$ on $S$, there is an open covering $\{S_\alpha\}$ of $S$ so that $E|_{S_\alpha}$ is trivial, so the map from $S$ to $U$ must map each $S_\alpha$ to $p$. Thus all of $S$ must map to $p$, which would force the bundle $E$ to be trivial.

The point is that the glue used to construct a vector bundle $E$ from trivial bundles $E_\alpha = \mathbb{C}^n \times S_\alpha$ on an open covering $\{S_\alpha\}$ is far from trivial. The gluing isomorphism from $E_\alpha|_{S_\alpha \cap S_\beta}$ to $E_\beta|_{S_\alpha \cap S_\beta}$ is given by a continuous map $\varphi_{\alpha\beta} \colon S_\alpha \cap S_\beta \to \mathrm{GL}_n(\mathbb{C})$. (Here we have $\mathrm{GL}_n(\mathbb{C})$ acting on the right on $\mathbb{C}^n$.) This gluing data must satisfy the compatibility

$$\varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \cdot \varphi_{\beta\gamma} \quad \text{on} \quad S_\alpha \cap S_\beta \cap S_\gamma.$$

(This forces $\varphi_{\alpha\alpha} = 1$ and $\varphi_{\beta\alpha} = \varphi_{\alpha\beta}^{-1}$.) From this one constructs a vector bundle $E$ on $S$, with isomorphisms $\varphi_\alpha \colon E|_{S_\alpha} \xrightarrow{\sim} E_\alpha$, so that $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ as maps from $E_\alpha|_{S_\alpha \cap S_\beta}$ to $E_\beta|_{S_\alpha \cap S_\beta}$.

If we set $T = \coprod S_\alpha$, the $\varphi_{\alpha\beta}$'s give a map $\Phi \colon T \times_X T \to \mathrm{GL}_n(\mathbb{C})$, and the compatibility says that this is a morphism of topological groupoids from $T \times_X T \rightrightarrows T$ to $\mathrm{GL}_n(\mathbb{C}) \rightrightarrows \bullet$. Thus we have a morphism from an atlas for $S$ to an atlas for $B\mathrm{GL}_n(\mathbb{C})$. (Note that giving a vector bundle of rank $n$ on $S$ is essentially the same as giving a principal $\mathrm{GL}_n(\mathbb{C})$-bundle on $S$.)

This construction of a bundle $E$ from the "cocycle" $\{\varphi_{\alpha\beta}\}$ is another example of *descent*, with $\{\varphi_{\alpha\beta}\}$ being the *descent data*.

EXERCISE 1.17. If vector bundles $E$ and $F$ arise from $\{\varphi_{\alpha\beta}\}$ and $\{\psi_{\alpha\beta}\}$ (on the same open covering), then $E$ is isomorphic to $F$ exactly when there are continuous maps $\vartheta_\alpha \colon S_\alpha \to \mathrm{GL}_n(\mathbb{C})$ such that

$$\vartheta_\alpha \cdot \psi_{\alpha\beta} = \varphi_{\alpha\beta} \cdot \vartheta_\beta \quad \text{on} \quad S_\alpha \cap S_\beta.$$

This data gives a map $\vartheta \colon T \to \mathrm{GL}_n(\mathbb{C})$ such that the two maps $(\vartheta \circ s) \cdot \Psi$ and $\Phi \cdot (\vartheta \circ t)$ from $T \times_S T$ to $\mathrm{GL}_n(\mathbb{C})$ agree.

Let us compare this to what is done in topology. Here there is a kind of universal bundle $E_U \to U$, with the property that every vector bundle on a (paracompact) space $S$ is isomorphic to $\varphi^*(E_U)$ for some map $\varphi \colon S \to U$. But $\varphi$ is not unique: homotopic maps $\varphi$ and $\psi$ determine isomorphic bundles. Modern topology depends not only on spaces and continuous maps between them, but also on this extra level of homotopies between maps. The map $\vartheta$ of the preceding exercise plays an analogous role in the world of groupoids.

In category theory there is a similar notion of a *natural transformation* between two functors from one category to another. These will play a role analogous to, and as important as, the role of homotopy in topology. A category with such an extra structure, called 2-*morphisms*, satisfying certain natural axioms, is called a 2-*category*. This implies in particular that, for objects $X$ and $Y$, the maps $\mathrm{Hom}(X, Y)$ between them form more than a set; it is actually itself a category. Stacks will form a 2-category, and so will groupoids. (The $\vartheta$ constructed in the preceding exercise is an example of a 2-morphism from $\Phi$ to $\Psi$.) Topologists have gotten used to working with homotopies, without needing the language of 2-categories. Those working with stacks are in a similar position, and the formalism of 2-categories is not required, as long as we are equally alert for natural isomorphisms. The reader can find the definition and basic properties of 2-categories in Appendix B.

## 8. Functors, fibers, and groupoids

Given a functor $\mathcal{X} \to \mathcal{S}$, and an object $S$ in $\mathcal{S}$, one has a category whose objects are the objects of $\mathcal{X}$ whose image in $\mathcal{S}$ is $S$, and whose morphisms are the morphisms in $X$ whose image in $\mathcal{S}$ is the identity map on $S$. This category is called the *fiber* of $\mathcal{X}$ over $S$, and is denoted $\mathcal{X}_S$.

For $\mathcal{X} = \underline{X}$, $\mathcal{X}_S$ is just the set of morphisms from $S$ to $X$; the only morphisms in $\mathcal{X}_S$ are identity maps.

For $\mathcal{X} = \mathcal{M}_g$, $\mathcal{X}_S$ has as objects the families $C \to S$ over $S$; its morphisms from $C' \to S$ to $C \to S$ are isomorphisms $C' \to C$ over $S$. And similarly for $BG$ or $[U/G]$. In general, the category $\mathcal{X}_S$ will be equivalent to the category of morphisms from $S$ (regarded as a stack) to $\mathcal{X}$.

Note that in every case, all morphisms in the category $\mathcal{X}_S$ are isomorphisms; that is, as a category, $\mathcal{X}_S$ is a groupoid. This will be required of any stack: $\mathcal{X} \to \mathcal{S}$ will be a *category fibered in groupoids*. (It is in this context that groupoids as categories appear in stack theory.) The more (non-identity) morphisms these categories $\mathcal{X}_S$ have, the farther the stack is from being a scheme. In fact, we will see that the algebraic stacks for which all fibers $\mathcal{X}_S$ are just sets are exactly the *algebraic spaces*, studied in Chapter 9. In short, sets are to groupoids as algebraic spaces are to stacks.

Another feature seen in our examples, and a crucial property of stacks, is that there is a *pullback* functor $f^*$ from $\mathcal{X}_S$ to $\mathcal{X}_T$ for every morphism $f\colon T \to S$ in $\mathcal{S}$. In many examples, the pullback of a family $X \to S$ by $f\colon T \to S$ in $\mathcal{S}$ is realized by the fibered product $X \times_S T \to T$.

A fundamental insight of Grothendieck, as we have seen, was to replace a scheme $X$ by its functor $h_X$ from schemes to sets. In this case the set $h_S(X)$ can be identified with the category $\mathcal{X}_S$, as this category has only identity maps as morphisms. In general, however, the assignment $S \rightsquigarrow \mathcal{X}_S$ is not quite a functor from $\mathcal{S}$ to the category (Gpds) of groupoids; it is only a "pseudofunctor". This problem appears in most of our examples, essentially because whenever we construct pullbacks $f^*$, it is rarely the case that $(f \circ g)^*$ is equal to $g^* \circ f^*$ for composable maps $f$ and $g$; at best, one has a canonical isomorphism between them. As in the examples described so far, most stacks can be defined without the language of pseudofunctors and the complications of these cocycle conditions. However, pseudofunctors provide a convenient way to think about stacks, and we include a discussion in Chapter 2 and Appendix B.

Before the theory of stacks was developed, an approach to moduli problems was to consider a functor $h$ from the base category $\mathcal{S}$ to the category of sets, with $h(S)$ being the set of isomorphism classes of families over $S$. For example, for $\mathcal{M}_g$, $h(S)$ was the set of families of curves $C \to S$, modulo isomorphism ([**75**]). For a morphism $f\colon S' \to S$, $f^*\colon h(S) \to h(S')$ is given by pullback. The functor $h$ is *representable* if there is an $X$ in $\mathcal{S}$, with $\xi$ in $h(X)$, such that every $\zeta$ in every $h(S)$ has the form $f^*(\xi)$ for a unique morphism $f\colon S \to X$. We say that $X$ *represents* $h$; the functors $h$ and $h_X$ are naturally isomorphic; one says that $X$ is a *fine* moduli space. One of the best known and most important examples of this is the functor that assigns to a scheme $S$ the set of closed subschemes of $\mathbb{P}^n \times S$, flat over $S$; this is represented by a Hilbert scheme.

Most such functors $h$, such as the one for moduli of curves, are not representable. As in this case, one often must settle for the existence of a *coarse* moduli space for $h$. This is an object $M$ with a natural transformation $h \to h_M$, with the university property that for any object $N$ and natural transformation $h \to h_N$, there is a unique morphism $M \to N$ so the induced $h_M \to h_N$ makes the diagram

$$\begin{array}{ccc} h & \longrightarrow & h_M \\ & \searrow & \downarrow \\ & & h_N \end{array}$$

commute; this universal property determines $M$, when it exists, up to canonical isomorphism. One often also requires that the map $h(p) \to h_M(p)$ be bijective when $p$ is a point (cf. [**75**], §5.2). There will be a similar notion of coarse moduli spaces for stacks.

## 9. Gluing and descent

The idea of gluing simple "local" objects together to form a global object goes back at least to the origins of manifolds. We have seen how this gluing data can be encoded in an atlas (a groupoid). In algebraic geometry, the simple objects are usually affine varieties or schemes. The notion of ringed spaces gave Serre and Grothendieck a good language for defining global varieties and schemes, and to carry out this gluing.

When $X$ is a scheme, the functor $h_X$ has a simple gluing property: if $\{S_\alpha\}$ is an open covering of $S$, a collection of morphisms $f_\alpha \colon S_\alpha \to X$, such that $f_\alpha = f_\beta$ on $S_\alpha \cap S_\beta$ for all $\alpha$ and $\beta$, determines a unique morphism $f \colon S \to X$. This is encoded by saying that

$$h_X(S) \to \prod_\alpha h_X(S_\alpha) \rightrightarrows \prod_{\alpha,\beta} h_X(S_\alpha \cap S_\beta)$$

is *exact*, or that $h_X$ is a *sheaf*.

If we think of a stack $\mathcal{X} \to \mathcal{S}$ as a kind of functor, taking $S$ to $\mathcal{X}_S$, the analogous property is considerably harder to describe: it is not so easy to glue categories. However, we will see that there is an analogous notion, which comes under the rubric of *descent theory*, a few instances of which we have already mentioned. To be a stack, a functor $\mathcal{X} \to \mathcal{S}$ will be required to satisfy an appropriate gluing/sheaf/descent property.

Grothendieck's representability theorem (cf. Glossary, [**EGA** I′], [**53**]) states that any contravariant functor from schemes to sets which is a sheaf in the Zariski topology, and which is covered by open subfunctors (suitably defined) each of which is representable by a scheme, is itself representable by a scheme. This says that one does not enlarge the category of schemes by gluing, if the Zariski topology is used. In fact, if one uses instead the étale topology, then a similar construction leads to the notion of an algebraic space.

In practice one usually glues simple objects — open subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$ in differential or analytic geometry, affine schemes in algebraic geometry. For example, if $X$ is a scheme, the functor $h_X$ is determined by its restriction to affine schemes, and this restriction captures both the construction of general schemes from affine schemes and the replacing of a scheme by a functor. In the same spirit, Laumon and Moret-Bailly [**61**] take their base category $\mathcal{S}$ to be the category of affine schemes. We do not do this here, in order to have a richer base category, but there is no essential difference in the resulting theory (see Chapter **??**).

The conditions put on a functor $\mathcal{X} \to \mathcal{S}$ to be an algebraic stack will guarantee that it has an atlas $R \rightrightarrows U$, with the projections $s$ and $t$ étale in the case of a Deligne-Mumford stack, and smooth in the case of an Artin stack. Such an atlas can itself be regarded as explicit and geometric gluing data for the stack. Constructing an atlas, or proving that it exists, is where the work must be done.

## Answers to Exercises

**1.2.** (a) $D \times S^1 \to X$, $(z, e^{i\vartheta}) \mapsto (z, \vartheta)$ is a 2-sheeted covering map, so satisfies the surjectivity requirement. (b) On $X$ the group $\mathrm{Aut}(x)$ is trivial for $x$ not on the central line, and it is $\{\pm 1\}$ for $x = (0, \vartheta)$.

**1.6.** For the category $\mathcal{T}$, take the same objects, but for morphisms allow the induced maps on fibers to be isometries followed by homotheties (multiplications by a positive scalar). Replace $\widetilde{T}$ by its intersection with the plane $a+b+c=1$, and enlarge $G$ by allowing homotheties. The resulting stack has dimension 2.

**1.7.** Under the substitution $\tau^2=\lambda$, sections correspond to maps $g\colon \mathbb{A}^1 \smallsetminus \{0\} \to E$ satisfying $g(-\tau)=-g(\tau)$, so sections correspond to 2-torsion points of $E$.