

MATH 116 — PRACTICE FOR EXAM 1

Generated January 5, 2023

NAME: SOLUTIONS

INSTRUCTOR: _____

SECTION NUMBER: _____

1. This exam has 4 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2017	1	5	counting money	12	
Winter 2018	1	2	owl and dove	9	
Winter 2016	1	5		16	
Winter 2018	1	6		7	
Total				44	

Recommended time (based on points): 40 minutes

5. [12 points] Several high-ranking Illuminati officials are relaxing while counting their money. There is so much money to count that the process takes many hours. The rate (in millions of dollars per hour) at which they count the money is given by the function $M(t)$, where t is the number of hours since they began counting. Several values of this function are given in the table below.

t (hours)	0	2	4	6	10	14
$M(t)$ (million \$/hour)	5	6	11	12	10	3

Note: The function $M(t)$ is continuous. Between each of the values of t given in the table, the function $M(t)$ is always increasing or always decreasing.

- a. [2 points] Write, but do **not** evaluate, a definite integral that gives the total amount of money, in millions of dollars, the officials counted from the time they started until the time when they were counting the money the fastest.

Solution: $\int_0^6 M(t) dt$

- b. [3 points] Write out the terms of a left Riemann sum with 3 equal subdivisions to estimate the integral from (a). Does this sum give an overestimate or an underestimate of the integral?

Solution: This left Riemann sum is $2 \cdot 5 + 2 \cdot 6 + 2 \cdot 11 = 44$.

Since $M(t)$ is increasing on the interval $[0, 6]$, this is an underestimate of the integral from (a).

- c. [4 points] Based on the data provided, write a sum that gives the best possible **overestimate** for the total amount of money, in millions of dollars, counted during the first 14 hours of counting.

Solution: The total amount of money, in millions of dollars, counted during the first 14 hours of counting is equal to $\int_0^{14} M(t) dt$.

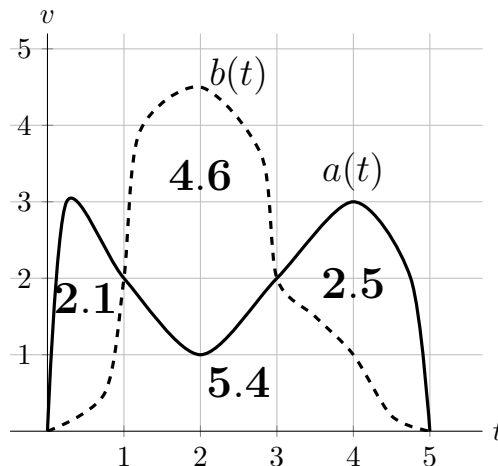
The best possible overestimate of this integral is given by the Riemann sum that uses right endpoints on the intervals over which f is increasing and left endpoints on the intervals over which f is decreasing. This gives the estimate $2 \cdot (6 + 11 + 12) + 4 \cdot (12 + 10)$.

- d. [3 points] What is the difference, in millions of dollars, between the best possible overestimate and the best possible underestimate for the total amount of money counted during the first 14 hours of counting?

Solution: We calculate the difference between the left and right hand sums on the intervals over which f is increasing and decreasing respectively and add these together, giving a difference of $2|12 - 5| + 4|3 - 12| = 50$.

2. [9 points]

When Alejandra and Brontel were children they spent summer mornings chasing birds in flight. One memorable day they encountered an owl. The following graph shows the velocities $a(t)$ of Alejandra (solid) and $b(t)$ of Brontel (dashed), measured in meters per second, t seconds after the owl took off. The area of each region is given.



- a. [1 point] How far (in meters) do Alejandra and Brontel chase the owl?

Solution: Summing the areas under either curve gives a total distance of 10 m.

- b. [5 points] Suppose the owl ascends to a height of h meters according to $h(t) = \sqrt{t}$ where t is seconds since it went airborne. Let $L(h)$ be the number of meters Alejandra is ahead of Brontel when the owl is h meters above ground. Write an expression for $L(h)$ involving integrals and compute $L'(2)$.

Solution: The owl is h meters above the ground at time $t = h^2$. Thus,

$$L(h) = \int_0^{h^2} a(t) - b(t) dt.$$

We compute $L'(h)$ using the second fundamental theorem of calculus.

$$L'(h) = 2ha(h^2) - 2hb(h^2).$$

So we have

$$L'(2) = 2 \cdot 2a(4) - 2 \cdot 2b(4) = 4 \cdot 3 - 4 \cdot 1 = 8,$$

where we get the values of $a(2)$ and $b(2)$ from the graph.

- c. [3 points] The next bird to pass is a dove. This time Alejandra runs twice as fast and Brontel runs three times as fast as they did when chasing the owl. How much faster (in m/s) is Brontel than Alejandra on average in the first 5 seconds?

Solution: The integral that represents this average is

$$\frac{1}{5} \int_0^5 3b(t) - 2a(t) dt = \frac{3}{5} \int_0^5 b(t) dt - \frac{2}{5} \int_0^5 a(t) dt.$$

Each of these integrals is equal to 10 as we see from the graph. Hence

$$\frac{1}{5} \int_0^5 3b(t) - 2a(t) dt = \frac{10}{5} = 2.$$

Answer: 2 m/s

5. [16 points] Suppose that $f(x)$ is a function with the following properties:

- $\int_0^1 f(x) dx = -5$.
- $\int_0^3 f'(x) dx = 10$.
- The average value of $f(x)$ on $[1, 1.5]$ is -4 .
- $\int_2^4 x f'(x) dx = 8$.

In addition, a table of values for $f(x)$ is given below.

x	0	1	2	3	4
$f(x)$	-7	-2	-2	m	0

Calculate (a)-(d) **exactly**. Show your work and do not write any decimal approximations.

a. [4 points] $m = 3$

Solution: Using the Fundamental Theorem in $\int_0^3 f'(x) dx = 10$ we get $f(3) - f(0) = 10$ which gives $m - (-7) = 10$ so $m = 3$.

b. [4 points] $\int_0^{1.5} f(x) dx = -7$

Solution:

$$\int_0^{1.5} f(x) dx = \int_0^1 f(x) dx + \int_1^{1.5} f(x) dx = -5 + 0.5(-4) = -7$$

c. [4 points] $\int_2^4 f(x) dx = -4$

Solution: Using integration by parts in $\int_2^4 x f'(x) dx = 8$ we get $(4f(4) - 2f(2)) - \int_2^4 f(x) dx = 8$ which gives $\int_2^4 f(x) dx = 0 - 2(-2) - 8 = -4$.

d. [4 points] $\int_4^{16} f'(\sqrt{x}) dx = 16$

Solution: Using the substitution $u = \sqrt{x}$ we get

$$\int_4^{16} f'(\sqrt{x}) dx = \int_2^4 f'(u) \cdot 2u du = 2 \cdot 8 = 16$$

6. [7 points] For each of the questions below, circle **all** of the available correct answers. You must circle at least one choice to receive any credit. No credit will be awarded for unclear markings. No justification is necessary.

- a. [3 points] Which $F(x)$ are antiderivatives of $f(x) = e^{x^2}$ with $F(3) = 5$ for $x > 0$?
 Note: due to a typo in the original exam (corrected here), a student's answer to option IV did not impact their score.

I. $F(x) = \int_0^{x^2} e^u du + 5$

II. $F(x) = \int_3^x 5e^{u^2} du$

III. $F(x) = \frac{1}{x^2} e^{x^2} + 5$

IV. $F(x) = \int_{x^2}^9 -\frac{1}{2\sqrt{u}} e^u du + 5$

V. $F(x) = \int_3^x e^{u^2} du + 5$

VI. $F(x) = \frac{e^{x^2}}{2x} - \frac{e^9}{6} + 5$

- b. [2 points] Suppose $f(x)$ is an odd function. Which values of b make the following equation true?

$$\int_{-\pi}^b \sin(f(x)) dx = 0$$

I. $b = -\pi$

II. $b = 0$

III. $b = \pi$

IV. $b = \frac{3\pi}{2}$

V. $b = 2\pi$

- c. [2 points] Which of the following could be the graph of $f(x) = \int_x^{x^3} e^{\sqrt[3]{u}} du$?

