Math 116 — Practice for Exam 2

Generated October 8, 2018

Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 8 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2015	2	6		15	
Winter 2014	2	11		10	
Winter 2015	2	9		10	
Fall 2015	2	1		12	
Fall 2014	2	5		10	
Winter 2018	3	4		11	
Winter 2016	2	1		14	
Fall 2016	2	11		12	
Total				94	

Recommended time (based on points): 88 minutes

6. [15 points] For each of the following questions, fill in the blank with the letter corresponding to the answer from the bottom of the page that correctly completes the sentence. No credit will be given for unclear answers. You do not need to show your work.

a. [3 points] The limit, $\lim_{x\to\infty}\left(\frac{x+2}{x}\right)^{x/2}$,...

B

b. [3 points] The value of the integral $\int_{-1}^{1}\frac{e}{x^{1/3}}dx$...

G

c. [3 points] The value of the integral $\int_{-1}^{2}\frac{8e}{x^3}dx$...

J or I

d. [3 points] The value of A for which the differential equation $y'' = Ay$ is satisfied by the function $f(t) = e^{et}$...

C

e. [3 points] The length of the polar curve $r = \frac{4e}{\pi}\cos(\theta)$ between $\theta = -\pi/4$ and $\theta = \pi/4$...

D

(A)...is $e^{1/2}$.	(F)...is 1.
(B)...is e.	(G)...is 0.
(C)...is e^2.	(H)...is 2.
(D) ...is $2e$.	(I)...does not exist.
(E)...is $3e$.	(J)...diverges.

11. [10 points]

a. [5 points] Compute the improper integral $\displaystyle\int_0^1 \ln(x)\, dx$. Show your work.

Solution:
$\int_0^1 \ln(x)\, dx = \lim_{a\to 0^+} \int_a^1 \ln(x)\, dx = \lim_{a\to 0^+} x\ln(x)|_a^1 - \int_a^1 1\, dx = \lim_{a\to 0^+} -a\ln(a) - 1 + a$. Using either L'hopital's rule or the fact that polynomials dominate logarithms we have $\lim_{a\to 0^+} a\ln(a) = 0$. Therefore the integral is equal to -1.

b. [5 points] Use comparison of improper integrals to determine if the improper integral $\displaystyle\int_1^\infty \frac{\sin(x)+3}{x^2+2}$ converges or diverges. Show your work.

Solution:
We have the inequalities $\sin(x)+3 \le 4$ and $\frac{1}{x^2+2} \le \frac{1}{x^2}$. Therefore $\int_1^\infty \frac{\sin(x)+3}{x^2+1}\, dx \le \int_1^\infty \frac{4}{x^2}\, dx = 4\int_1^\infty \frac{1}{x^2}\, dx$. This integral is a p-integral with $p = 2 > 1$ so it converges. Therefore $\int_1^\infty \frac{\sin(x)+3}{x^2+2}\, dx$ converges by comparison.

9. [10 points] Determine if the following integrals converge or diverge. If the integral converges, circle the word "converges" and give the **exact value** (i.e. no decimal approximations). If the integral diverges, circle "diverges". In either case, **you must show all your work and indicate any theorems you use**. Any direct evaluation of integrals must be done **without using a calculator.**

a. [5 points] $\displaystyle\int_0^1 \ln(x)\,dx$

$\boxed{\text{CONVERGES}}$ DIVERGES

Solution: We have

$$\begin{aligned}\int_0^1 \ln(x)dx &= \lim_{a\to 0+}\int_a^1 \ln(x)dx \\ &= \lim_{a\to 0+} x\ln(x) - x\big|_a^1 \\ &= \lim_{a\to 0+} -1 - a\ln(a) + a.\end{aligned}$$

L'Hospital's Rule tells us that $\lim_{a\to 0+} a\ln(a) = 0$. Thus, $\displaystyle\int_0^1 \ln(x)dx = -1$

b. [5 points] $\displaystyle\int_2^\infty \frac{x+\sin x}{x^2-x}\,dx$

CONVERGES $\boxed{\text{DIVERGES}}$

Solution: We have

$$\begin{aligned}\sin(x) &\geq -1, \\ x+\sin(x) &\geq x-1, \\ \frac{x+\sin(x)}{x^2-x} &\geq \frac{x-1}{x^2-x} = \frac{1}{x}.\end{aligned}$$

We also know that $\displaystyle\int_2^\infty \frac{1}{x}dx$ diverges, as it is an improper integral of the form $\displaystyle\int_a^\infty \frac{1}{x^p}dx$ for $p \leq 1$. Thus, by the comparison test we have that $\displaystyle\int_2^\infty \frac{x+\sin x}{x^2-x}dx$ diverges.

1. [12 points] Determine if the following integrals converge or diverge. If the integral converges, circle the word "converges" and give the **exact value** (i.e. no decimal approximations). If the integral diverges, circle "diverges". In either case, **you must show all your work and indicate any theorems you use**. Any direct evaluation of integrals must be done **without using a calculator**.

a. [6 points] $\displaystyle\int_0^1 \frac{e^x\sin(2x)-(2e^x-2)\cos(2x)}{\sin^2(2x)}\,dx$

$$\left(\text{Note: } \frac{d}{dx}\left(\frac{e^x-1}{\sin(2x)}\right) = \frac{e^x\sin(2x)-(2e^x-2)\cos(2x)}{\sin^2(2x)}\right)$$

$\boxed{\textbf{Converges}}$ Diverges

Solution:

$$\begin{aligned}\int_0^1 \frac{e^x\sin(2x)-(2e^x-2)\cos(2x)}{\sin^2(2x)}\,dx &= \lim_{a\to 0^+}\int_a^1 \frac{e^x\sin(2x)-(2e^x-2)\cos(2x)}{\sin^2(2x)}\,dx\\ &= \lim_{a\to 0^+}\left[\frac{e^x-1}{\sin(2x)}\right]_a^1\\ &= \lim_{a\to 0^+}\left(\frac{e-1}{\sin(2)}-\frac{e^a-1}{\sin(2a)}\right)\\ &= \frac{e-1}{\sin(2)}-\lim_{a\to 0^+}\left(\frac{e^a}{2\cos(2a)}\right) \quad \text{by L'Hopital's Rule}\\ &= \frac{e-1}{\sin(2)}-\frac{1}{2}.\end{aligned}$$

b. [6 points] $\displaystyle\int_2^\infty \frac{1}{(\ln x)^2 x}\,dx$

$\boxed{\textbf{Converges}}$ Diverges

Solution:

$$\int_2^\infty \frac{1}{(\ln x)^2 x}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{(\ln x)^2 x}\,dx$$

Using the substitution $w=\ln x, dw=\frac{1}{x}dx$, the integral becomes

$$\begin{aligned}\lim_{b\to\infty}\int_{\ln 2}^{\ln b}\frac{1}{w^2}\,dw &= \lim_{b\to\infty}\left[\frac{-1}{w}\right]_{\ln 2}^{\ln b}\\ &= \lim_{b\to\infty}\left(\frac{-1}{\ln b}-\frac{-1}{\ln 2}\right)\\ &= \frac{1}{\ln 2}\end{aligned}$$

5. [10 points] Determine whether the following integrals converge or diverge. If the integral converges, find the exact value. You must show all work, and perform any integral computations by hand.

a. [5 points] $\displaystyle\int_2^\infty \frac{dx}{x(\ln(x))^2}$

Solution: We start with the substitution $u = \ln(x)$:

$$\int_2^\infty \frac{dx}{x(\ln(x))^2} = \int_{\ln(2)}^\infty \frac{du}{u^2}$$

Now we integrate normally:

$$\begin{aligned}
\int_{\ln(2)}^\infty \frac{du}{u^2} &= \lim_{a\to\infty} \int_{\ln(2)}^a \frac{du}{u^2} \\
&= \lim_{a\to\infty} -\frac{1}{u}\Big|_{\ln(2)}^a \\
&= \lim_{a\to\infty} -\frac{1}{a} + \frac{1}{\ln(2)} \\
&= \frac{1}{\ln(2)}
\end{aligned}$$

b. [5 points] $\displaystyle\int_1^3 \frac{x}{(x-3)^3}dx$

Solution: You can do this problem with integration by parts and by substitution. For substitution, use $u = (x-3)$, so the integral becomes:

$$\int_1^3 \frac{x}{(x-3)^3}dx = \int_{-2}^0 \frac{u+3}{u^3}du$$

This integral splits up as:

$$\begin{aligned}
\int_{-2}^0 \frac{3}{u^3} + \frac{1}{u^2}du &= \lim_{a\to 0}\int_{-2}^a \frac{3}{u^3} + \frac{1}{u^2}du \\
&= \lim_{a\to 0^-}\left(\frac{-3}{2u^2} - \frac{1}{u}\right)\Big|_{-2}^a \\
&= \lim_{a\to 0^-}\left(\frac{-3/2-u}{u^2}\right)\Big|_{-2}^a \\
&= \lim_{a\to 0^-}\left(\frac{-3/2-a}{a^2}\right) - \left(\frac{-3/2-(-2)}{(-2)^2}\right)
\end{aligned}$$

But $\displaystyle\lim_{a\to 0^-}\left(\frac{-3/2-a}{a^2}\right) = -\infty$, so the integral diverges. To solve this by parts, set $u = x$ and $dv = \frac{1}{(x-3)^3}$, so $du = dx$ and $v = \frac{-1}{(x-3)^2}$.

$$\begin{aligned}
\int_1^3 \frac{x}{(x-3)^3}dx &= \lim_{a\to 3^-}\left(\frac{-x}{(x-3)^2}\Big|_1^a - \int_1^a \frac{-1}{(x-3)^2}\right) \\
&= \lim_{a\to 3^-}\left(\frac{-x-(x-3)}{(x-3)^2}\Big|_1^a\right) \\
&= -\infty
\end{aligned}$$

4. [11 points]

a. [6 points] Determine whether the following series converges absolutely, converges conditionally, or diverges, and give a complete argument justifying your answer.

$$\sum_{n=1}^{\infty}(-1)^n \sin\left(\tfrac{1}{n}\right)$$

Converges absolutely $\boxed{\textbf{Converges conditionally}}$ **Diverges**

Justification:

Solution: This series converges by the alternating series test, which applies, since $\sin(\frac{1}{n})$ is a positive decreasing sequence that converges to zero.

It does not converge absolutely since for $n \geq 1$

$$\frac{1}{2n} \leq \sin\left(\tfrac{1}{n}\right).$$

We know the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n}$ diverges by p-test with $p = 1$. Then by the comparison test, so must $\displaystyle\sum_{n=1}^{\infty}\sin\left(\tfrac{1}{n}\right) = \sum_{n=1}^{\infty}\left|(-1)^n \sin\left(\tfrac{1}{n}\right)\right|$.

Alternatively, we can use the Limit Comparison Test. Since

$$\lim_{n\to\infty}\frac{\sin(1/n)}{1/n} = \lim_{x\to\infty}\frac{\sin(1/n)}{1/n} = \lim_{y\to 0}\frac{\sin(y)}{y} = 1 < 0,$$

we know that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty}\sin\left(\tfrac{1}{n}\right)$ must either both converge or both diverge. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is the harmonic series, which we know diverges, $\displaystyle\sum_{n=1}^{\infty}\sin\left(\tfrac{1}{n}\right)$ must diverge as well.

b. [5 points] Compute the value of the following improper integral. **Show all your work using correct notation.** Evaluation of integrals must be done **without a calculator.**

$$\int_0^\infty \frac{e^x}{(1+e^x)^2}\,dx$$

Solution:
First we change to limit notation, then use u-substitution with $u = 1 + e^x$.

$$\begin{aligned}
\int_0^\infty \frac{e^x}{(1+e^x)^2}\,dx &= \lim_{b\to\infty}\int_0^b \frac{e^x}{(1+e^x)^2}\,dx \\
&= \lim_{b\to\infty}\int_2^b \frac{1}{u^2}\,du \\
&= \lim_{b\to\infty} -\frac{1}{u}\Big|_2^b \\
&= \lim_{b\to\infty} \frac{-1}{b} - \frac{-1}{2} = \frac{1}{2}
\end{aligned}$$

Alternatively, first compute the antiderivative using u-substitution.

$$\int \frac{e^x}{(1+e^x)^2}\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} = -\frac{1}{1+e^x}.$$

Thus,

$$\int_0^\infty \frac{e^x}{(1+e^x)^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{e^x}{(1+e^x)^2}\,dx = \lim_{b\to\infty} -\frac{1}{1+e^b} + \frac{1}{2} = \frac{1}{2}.$$

1. [14 points] Determine if the following integrals converge or diverge. If the integral converges, circle the word "converges" and give the **exact value** (i.e. no decimal approximations). If the integral diverges, circle "diverges". In either case, **you must give full evidence supporting your answer, showing all your work and indicating any theorems about improper integrals you use**. Any direct evaluation of integrals must be done **without using a calculator**.

a. [7 points] $\displaystyle\int_1^\infty \frac{x}{e^{ax^2+1}}\,dx$, where $a > 0$ is a constant

$\boxed{\textbf{Converges}}$ Diverges

Solution:

$$\int_1^\infty \frac{x}{e^{ax^2+1}}\,dx = \lim_{b\to\infty}\int_1^b \frac{x}{e^{ax^2+1}}\,dx = \lim_{b\to\infty}\int_{a+1}^{ab^2+1} \frac{1}{2ae^u}\,du =$$

$$= \lim_{b\to\infty}\left(\frac{-e^{ab^2+1}}{2a} - \frac{-e^{a+1}}{2a}\right) = \frac{e^{a+1}}{2a} = \frac{1}{2ae^{a+1}}$$

In the second equality, we used the substitution $u = ax^2 + 1$.

b. [7 points] $\displaystyle\int_2^\infty \frac{x+\sin x}{x^2}\,dx$

Converges $\boxed{\textbf{Diverges}}$

Solution: Since $\sin x \geq -1$ for any x,

$$\frac{x + \sin x}{x^2} \geq \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2} \quad (*)$$

The improper integral

$$\int_2^\infty \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$$

diverges because the improper integral

$$\int_2^\infty \frac{1}{x}\, dx$$

diverges ($p = 1 \leq 1$) while the improper integral

$$\int_2^\infty \frac{1}{x^2}\, dx$$

converges ($p = 2 > 1$). So using ($*$), the integral in question diverges by the comparison test.

Alternatively, we can use the inequality

$$\frac{x + \sin x}{x^2} \geq \frac{x-1}{x^2} \geq \frac{1}{2x}$$

which is valid for all $x \geq 2$, the p-test and the comparison test.

11. [12 points] Determine whether the following integrals converge or diverge. If an integral converges, find its exact value (i.e., no decimal approximations) and write it in the blank provided. If it diverges, circle "DIVERGES" and explain why. In any case, **show all your work, indicating any theorems you use, and using proper syntax and notation**.

a. [6 points] $\displaystyle\int_0^\infty 2xe^{-cx}\,dx$, where $c > 0$ is a constant

DIVERGES **CONVERGES TO** $2/c^2$

Solution: For $b \geq 0$ we have

$$\int_0^b xe^{-cx}\,dx = -\frac{xe^{-cx}}{c}\bigg|_0^b + \frac{1}{c}\int_0^b e^{-cx}\,dx = -\frac{be^{-cb}}{c} + \frac{1-e^{-cb}}{c^2},$$

so

$$\int_0^\infty 2xe^{-cx}\,dx = 2\lim_{b\to\infty}\left(-\frac{be^{-cb}}{c} + \frac{1-e^{-cb}}{c^2}\right) = \frac{2}{c^2}.$$

b. [6 points] $\displaystyle\int_0^1 \frac{x}{\sqrt{x^5+x^7}}\,dx$

$\boxed{\textbf{DIVERGES}}$ **CONVERGES TO** __________

Solution: For $0 < x \leq 1$ we have

$$\frac{x}{\sqrt{x^5+x^7}} \geq \frac{x}{\sqrt{2x^5}} = \frac{1}{\sqrt{2}x^{3/2}}.$$

Since

$$\frac{1}{\sqrt{2}}\int_0^1 \frac{dx}{x^{3/2}}$$

diverges by the p-Test with $p = \frac{3}{2}$, the original integral diverges by comparison.

Alternatively, notice that

$$\lim_{x\to 0^+}\frac{x/\sqrt{x^5+x^7}}{1/x^{3/2}} = 1.$$

Since

$$\int_0^1 \frac{dx}{x^{3/2}}$$

diverges by the p-Test with $p = \frac{3}{2}$, the original integral diverges by the Limit Comparison Test.