Math 116 — Practice for Exam 3

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Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 7 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2012	3	9		10	
Winter 2014	3	3		10	
Fall 2014	3	2		6	
Fall 2014	3	12	chicken coop	8	
Fall 2015	3	2		8	
Winter 2017	3	10		8	
Fall 2017	3	6		8	
Total				58	

Recommended time (based on points): 70 minutes

9. [10 points] A second way to approximate the function

$$F(x) = \int_0^x \sqrt{1+9t^4}dt.$$

is by using its Taylor polynomials.

a. [2 points] Find the first three nonzero terms in the Taylor series for the function $\sqrt{1+u}$ about $u = 0$.

Solution:

$$\sqrt{1+u} \approx 1 + \frac{1}{2}u - \frac{1}{8}u^2$$

b. [2 points] Find the first three nonzero terms in the Taylor series for $\sqrt{1+9t^4}$ about $t = 0$.

Solution: Let $u = 9t^4$ then

$$\sqrt{1+9t^4} \approx 1 + \frac{9}{2}t^4 - \frac{81}{8}t^8$$

c. [2 points] Find the first three nonzero terms in the Taylor series for $F(x)$ about $x = 0$.

Solution:

$$F(x) \approx \int_0^x 1 + \frac{9}{2}t^4 - \frac{81}{8}t^8 dt = x + \frac{9}{10}x^5 - \frac{9}{8}x^9$$

d. [2 points] For which values of x do you expect the Taylor series for $F(x)$ about $x = 0$ to converge? Justify your answer.

Solution: We substituted $u = 9t^4$ into the Binomial series. The interval of convergence for the Binomial series is $-1 < u < 1$. Then we expect the series to converge for $0 \leq 9x^4 < 1$. Hence the Taylor series for $F(x)$ about $x = 0$ converges if $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$.

e. [2 points] Use the fifth degree Taylor polynomial for $F(x)$ about $x = 0$ to approximate the value of $F(\frac{1}{2})$.

Solution: $P_5(x) = x + \frac{9}{10}x^5$, then $F\left(\frac{1}{2}\right) \approx P_5\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{9}{10}\left(\frac{1}{2}\right)^5 = 0.528$

3. [10 points] For each of the following questions circle the correct answer.

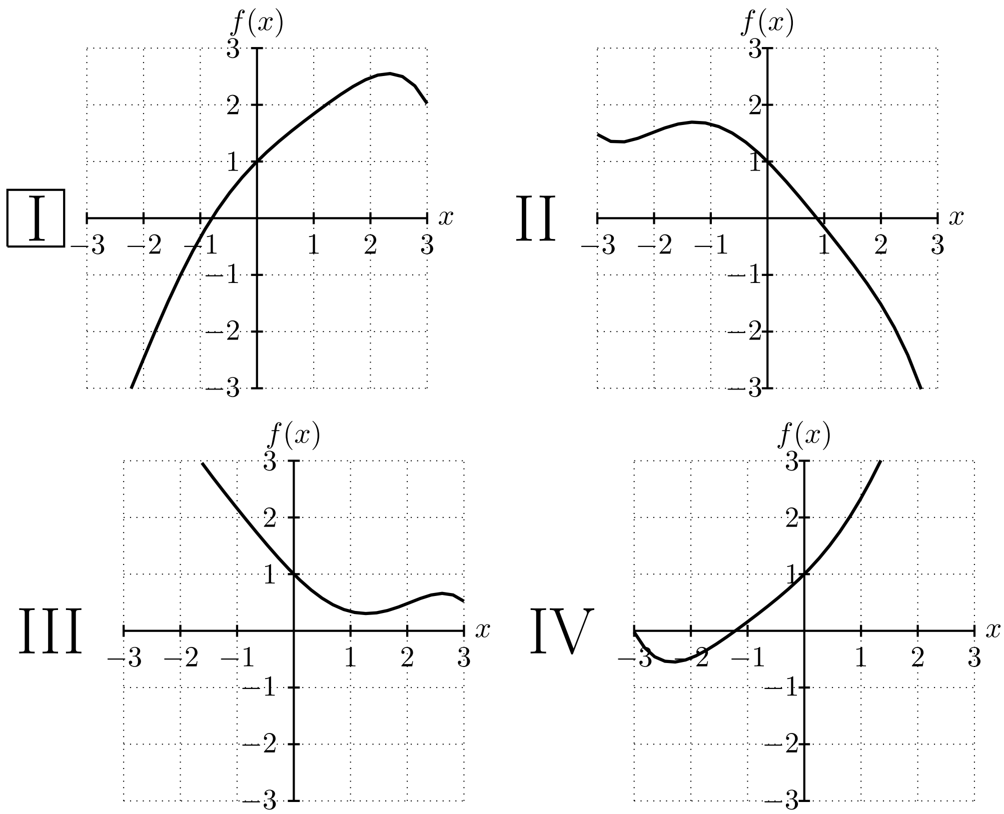
a. [2 points] What is the value of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{n!}$?

$\cos(2)$ $\quad$ e^{-2} $\quad$ $\cos(4)$ $\quad$ $\boxed{e^{-4}}$

b. [2 points] What is the value of the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n}(-1)^n}{(2n+1)!}$?

$\frac{1}{2}\sin(2)$ $\quad$ $\sin(2) - 2$ $\quad$ $\sin(2)$ $\quad$ $\boxed{\frac{1}{2}(\sin(2) - 2)}$

c. [2 points] Suppose that $1 + x - \frac{1}{4}x^2 + \frac{1}{10}x^3$ is the 3rd degree Taylor polynomial for a function $f(x)$. Which of the following pictures could be a graph of $f(x)$?



d. [2 points] What is the Taylor series of $2xe^{x^2}$ centered at $x = 0$?

$\boxed{\displaystyle\sum_{n=0}^{\infty} \frac{2x^{2n+1}}{n!}}$ $\quad$ $\displaystyle\sum_{n=1}^{\infty} \frac{2x^{2n-1}}{n!}$ $\quad$ $\displaystyle\sum_{n=1}^{\infty} \frac{2x^{2n+1}}{(n-1)!}$ $\quad$ $\displaystyle\sum_{n=0}^{\infty} \frac{2x^{2n-1}}{n!}$

e. [2 points] The radius of convergence of the Taylor series $\displaystyle\sum_{n=1}^{\infty} \frac{(x+5)^n 5^{-n}}{n+5}$ is $R = 5$. What is the interval of convergence of the series?

$\boxed{[-10, 0)}$ $\quad$ $(-10, 0)$ $\quad$ $(0, 10]$ $\quad$ $[-10, 0]$ $\quad$ $[0, 10)$

2. [6 points] Let $f(x) = xe^{-x^2}$.

a. [4 points] Find the Taylor series of $f(x)$ centered at $x = 0$. Be sure to include the first 3 nonzero terms and the general term.

Solution: We can use the Taylor series of e^y to find the Taylor series for e^{-x^2} by substituting $y = -x^2$.

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \cdots + \frac{(-x^2)^n}{n!} + \cdots$$

Therefore the Taylor series of xe^{-x^2} is

$$xe^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!} = x - x^3 + \frac{x^5}{2!} + \cdots + \frac{(-1)^n x^{2n+1}}{n!} + \cdots$$

b. [2 points] Find $f^{(15)}(0)$.

Solution: We know that $\frac{f^{(15)}(0)}{15!}$ will appear as the coefficient of the degree 15 term of the Taylor series. Using part (**a**), we see that the degree 15 term has coefficient $\frac{-1}{7!}$. Therefore

$$f^{(15)}(0) = \frac{-15!}{7!} = -259,459,200$$

3. [3 points] Determine the exact value of the infinite series

$$1 - \frac{2}{1!} + \frac{4}{2!} - \frac{8}{3!} + \cdots + \frac{(-1)^n 2^n}{n!} + \cdots$$

Solution: Notice that this is the Taylor series for e^y applied to $y = -2$. Therefore, the series has exact value e^{-2}.

12. [8 points] Franklin, your friendly new neighbor, is building a large chicken sanctuary. You decide to help Franklin build a special chicken coop with volume (in cubic km) given by the integral

$$\int_0^1 x\sqrt{1-\cos(x^2)}\,dx.$$

This integral is difficult to evaluate precisely, so you decide to use the methods you've learned this semester to help out Franklin. Your friend and president-elect, Kazilla, stops by to give you a hand. She suggests finding the 4th degree Taylor polynomial, $P_4(x)$, for the function $1-\cos(x^2)$ near $x=0$.

a. [4 points] Find $P_4(x)$.

Solution: We can use the Taylor series expansion for $\cos(x^2)$

$$\cos(y) = \sum_{n=0}^{\infty} \frac{(-1)^n\, y^{2n}}{(2n)!} = 1 - \frac{y^2}{2!} + \cdots + \frac{(-1)^n\, y^{2n}}{(2n)!} + \cdots$$

so

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n\, (x^2)^{2n}}{(2n)!} = 1 - \frac{(x^2)^2}{2!} + \cdots + \frac{(-1)^n\, (x^2)^{2n}}{(2n)!} + \cdots$$

Therefore

$$P_4 = 1 - (1 - \frac{x^4}{2!}) = \frac{x^4}{2!}$$

b. [4 points] Substitute $P_4(x)$ for $1-\cos(x^2)$ in the integral and compute the resulting integral by hand, showing all of your work.

Solution:

$$\begin{aligned}
\int_0^1 x\sqrt{P_4(x)}dx &= \int_0^1 x\sqrt{\frac{x^4}{2}} \\
&= \int_0^1 \frac{x^3}{\sqrt{2}} \\
&= \frac{x^4}{4\sqrt{2}}\Bigg|_{x=0}^{1} \\
&= \frac{1}{4\sqrt{2}}
\end{aligned}$$

2. [8 points] Let $f(x) = x^{2x}$. The first two derivatives of f are given below.

$$f'(x) = 2(1 + \ln x)x^{2x}$$
$$f''(x) = 2x^{2x-1} + 4(1 + \ln x)^2 x^{2x}$$

a. [4 points] Find the 2nd degree Taylor polynomial $P_2(x)$ of f centered at $x = 1$.

Solution: Using the formula for Taylor polynomials,

$$\begin{aligned} P_2(x) &= f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\ &= 1 + 2(x-1) + 3(x-1)^2 \end{aligned}$$

$P_2(x) =$ $1 + 2(x-1) + 3(x-1)^2$

b. [4 points] Find

$$\lim_{x \to 1} \frac{x^{2x} - 1}{3x - 3}.$$

Clearly show your reasoning. Your answer from part **(a)** may be helpful.

Solution:

$$\begin{aligned} \lim_{x \to 1} \frac{x^{2x} - 1}{3x - 3} &= \lim_{x \to 1} \frac{1 + 2(x-1) + 3(x-1)^2 - 1}{3(x-1)} \\ &= \lim_{x \to 1} \frac{2 + 3(x-1)}{3} \\ &= \frac{2}{3} \end{aligned}$$

10. [8 points] The Taylor series centered at $x = 0$ for a function $F(x)$ converges to $F(x)$ for all x and is given below.

$$F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n)!(4n+1)}$$

a. [3 points] What is the value of $F^{(101)}(0)$?
Make sure your answer is exact. You do not need to simplify.

$F(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} x^n$ by Taylor's Theorem. So the x^{101} term is $\frac{F^{(101)}(0)}{101!} x^{101}$. When you plug $n=25$ into the series above you get $-\frac{x^{101}}{50!(101)}$

So $F^{(101)}(0) = \frac{-101!}{50!(101)}$

Answer: $F^{(101)}(0) =$ $-\frac{100!}{50!}$

b. [3 points] Find $P_9(x)$, the 9th degree Taylor polynomial that approximates $F(x)$ near $x = 0$.

$$P_9(x) = \frac{x^{0+1}}{0!(0+1)} - \frac{x^{4+1}}{2!(4+1)} + \frac{x^{8+1}}{4!(8+1)}$$

$$= x - \frac{1}{10}x^5 + \frac{1}{216}x^9$$

c. [2 points] Use your Taylor polynomial from part **b.** to compute

$$\lim_{x \to 0} \frac{F(x) - x}{2x^5}$$

$$\lim_{x\to 0} \frac{F(x)-x}{2x^5} = \lim_{x\to 0} \frac{\left[x - \frac{1}{10}x^5 + \frac{1}{216}x^9\right] - x}{2x^5}$$

$$= \lim_{x\to 0} -\frac{1}{20} + \frac{1}{432}x^4 = -\frac{1}{20}$$

6. [8 points]

Values of a function f and some of its derivatives are given in the table on the right. Use this information to answer the questions that follow.

x	0	π
$f(x)$	-6	2π
$f'(x)$	6	2
$f''(x)$	1	-3
$f'''(x)$	-1	0
$f''''(x)$	5	$-9/2$

a. [4 points] Find a formula for the Taylor polynomial of degree 4 for f about $x = \pi$.

$$P_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}(\pi)}{n!}(x-\pi)^n$$

$$= f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2 + \frac{f'''(\pi)}{3!}(x-\pi)^3 + \frac{f^{IV}(\pi)}{4!}(x-\pi)^4$$

$$= 2\pi + 2(x-\pi) + \frac{-3}{2}(x-\pi)^2 + \frac{0}{6}(x-\pi)^3 + \frac{-9/2}{24}(x-\pi)^4$$

$$= \boxed{2\pi + 2(x-\pi) - \frac{3}{2}(x-\pi)^2 - \frac{3}{16}(x-\pi)^4}$$

b. [4 points] Find the first three nonzero terms of the Taylor series for $\int_0^x f(t^2)\,dt$ about $x = 0$.

Near 0, $f(x) \approx f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$

$$= -6 + 6x + \frac{1}{2}x^2$$

So $\int_0^x f(t^2)\,dt \approx \int_0^x -6 + 6t^2 + \frac{1}{2}t^4\,dt$

$$= -6t + 2t^3 + \frac{1}{10}t^5 \Big|_0^x$$

$$= \boxed{-6x + 2x^3 - \frac{1}{10}x^5}$$