Math 116 — Series Worksheet Solutions

Winter 2014

1. Determine whether the following series converge or diverge using the techniques you have learned in class.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+\sin(n)}{n^3+3}$

Here we can use the limit comparison test. Take $a_n = \frac{n^2+\sin(n)}{n^3+3}$ and $b_n = \frac{1}{n}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+\sin(n)}{n^3+3}\frac{n}{1} = \lim_{n\to\infty} \frac{n^3+n\sin(n)}{n^3+3} = 1$. Therefore the two series have the same behavior. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges so $\sum_{n=1}^{\infty} \frac{n^2+\sin(n)}{n^3+3}$ must also diverge.

by p-test $(p=1)$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{e^n}$

The fastest way to determine convergence here is to remember that factorials grow faster than exponentials. Therefore $\lim_{n\to\infty} \frac{n!}{e^n} = \infty$ so the series must diverge. by the n^{th} term test.

You can also use the ratio test. For the ratio test we consider $\lim_{n\to\infty} |\frac{(n+1)!e^n}{e^{n+1}n!}| = \lim_{n\to\infty} |\frac{(n+1)}{e}| = \infty$ so the series diverges by the ratio test.

(c) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n\ln(n)^2}$

False. Only if its $\sum \frac{1}{n\cdot(\ln n)^k}$, that is, just n, not a power of n.

The integral test is the best choice for series that are combinations of polynomials and logarithms. First we need to check that we can use the integral test. Take $f(x) = \frac{1}{x\ln(x)^2}$ then $f(x)$ is positive and decreasing so we may use the integral test. We get the integral $\int_4^{\infty} \frac{1}{x\ln(x)^2}dx$. Change variables setting $u = \ln(x)$. Then $du = \frac{1}{x}dx$. Thus we get the integral $\int_{\ln(4)}^{\infty} \frac{1}{u^2}du$. This integral converges by the p-test so the original series also converges. with $p=2$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^{n!}}$

Clearly $e^{n!} \geq e^n$. Therefore $\frac{1}{e^{n!}} \leq e^{-n}$. Using comparison $\sum_{n=0}^{\infty} \frac{1}{e^{n!}} \leq \sum_{n=0}^{\infty} e^{-n}$ which is a convergent geometric series with ratio $\frac{1}{e}$. Therefore $\sum_{n=0}^{\infty} \frac{1}{e^{n!}}$ converges by comparison test.

Or: Ratio Test

$$\lim_{n\to\infty} \frac{1/e^{(n+1)!}}{1/e^{n!}} = \lim_{n\to\infty} \frac{1}{e^{(n+1)!-n!}} = \lim_{n\to\infty} \frac{1}{e^{n!\cdot(n)}} = 0$$

so the series converges by the Ratio Test.

(e) $\displaystyle\sum_{n=1}^{\infty} \sin(n)$

$\lim_{n\to\infty} \sin(n)$ does not exist. ~~Since the terms aren't going to zero~~ The series diverges. by the n^{th} term test

(f) $\displaystyle\sum_{n=1}^{\infty} \tan(n)$

$\lim_{n\to\infty} \tan(n)$ does not exist. ~~Since the terms aren't going to zero~~ The series diverges. by the n^{th} term test

(g) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{6^n}$

This is a convergent geometric series with ~~$x =$~~ ratio $-1/6$. It is also possible to use the alternating series test to show that the series converges. We know that $\frac{1}{6^n}$ is decreasing and has limit zero. Therefore we may apply the alternating series test to $\sum_{n=0}^{\infty} \frac{(-1)^n}{6^n}$ which tells us that it converges.

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{n^2}}{n^3}$.

by the n^{th} term test

$\lim_{n\to\infty} \frac{e^{n^2}}{n^3} = \infty$ since exponentials grow fa s ter than powers. Therefore the series diverges by the n^{th} term test ~~since the terms don't go to zero~~.

(i) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^3+7}$

Could also use LCT: $1 = \lim_{n\to\infty} \frac{1/(n^3+7)}{1/n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ conv. (p-test, $p=3$) so $\sum_{n=0}^{\infty} \frac{1}{n^3+7}$ conv. by LCT.

$\sum_{n=0}^{\infty} \frac{1}{n^3+7} = \frac{1}{7} + \sum_{n=1}^{\infty} \frac{1}{n^3+7} \le \frac{1}{7} + \sum_{n=1}^{\infty} \frac{1}{n^3}$. We know that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by the p-test with $p=3$. Therefore the original series converges by the comparison test.

(j) $\displaystyle\sum_{n=1}^{\infty} \cos(\pi n)\ln(1+\frac{1}{n})$

The first thing to notice is that $\cos(\pi n) = (-1)^n$. This suggests that we should try to use the alternating series test. $\lim_{n\to\infty} \ln(1+\frac{1}{n}) = \ln(1) = 0$ and $\ln(1+\frac{1}{n})$ is decreasing in n. Therefore the alternating series test applies and therefore $\sum_{n=1}^{\infty} \cos(\pi n)\ln(1+\frac{1}{n})$ converges.

(k) $\displaystyle\sum_{n=1}^{\infty} \sin(1/n)$

OR: use LCT. $\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ div. (p-test, $p=1$) so $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$ div. by LCT.

$\sin(1/x)$ is positive for $x \ge 1$ and is also decreasing so we can use the integral test. We want to consider the integral $\int_1^{\infty} \sin(1/x)dx$. Changing variables $u = 1/x$ then $du = -\frac{1}{x^2}dx$ so $dx = -\frac{1}{u^2}du$. So we get the integral $\int_0^1 \frac{\sin(u)}{u^2}$. Now $\int_0^1 \frac{\sin(u)}{u^2} > \sin(1)\int_0^1 \frac{1}{u}du$. The final integral diverges by the p-test therefore $\int_1^{\infty} \sin(1/x)dx$ diverges. Thus the integral test tells us $\sum_{n=1}^{\infty} \sin(1/n)$ diverges.

(l) $\displaystyle\sum_{n=1}^{\infty} \cos(1/n)$

$\lim_{n\to\infty} \cos(1/n) = \cos(0) = 1$ therefore the terms of the series don't go to zero so the series diverges. by n^{th} term test.

(m) $\displaystyle\sum_{n=1}^{\infty} \sin(e^{-n})$

Use LCT: $\lim_{n\to\infty} \frac{\sin(e^{-n})}{e^{-n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\sum_{n=1}^{\infty} e^{-n}$ conv. (geom. ser., ratio $\frac{1}{e}$) so $\sum_{n=1}^{\infty} \sin(e^{-n})$ conv. by LCT.

Not taught in this course.

The graph of $y = \sin(x)$ is always below the graph of $y = x$ when $x \ge 0$. Therefore $\sin(e^{-n}) \le e^{-n}$. Therefore using comparison $\sum_{n=1}^{\infty} \sin(e^{-n}) \le \sum_{n=1}^{\infty} e^{-n}$. The second series is a convergent geometric series with ratio $\frac{1}{e}$. Therefore the original series converges by comparison test.

(n) $\displaystyle\sum_{n=1}^{\infty} (-1)^n e^{1/n}$

by n^{th} term test.

$\lim_{n\to\infty} (-1)^n e^{1/n} = \lim_{n\to\infty} (-1)^n$. The limit does not exist therefore the series diverges by n^{th} term test ~~since the terms don't go to zero~~.

(o) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{\ln(n)^{\ln(n)}}$ DON'T WORRY ABOUT THIS ONE - TOO TIME-CONSUMING FOR A TEST.

$\frac{1}{\ln(x)^{\ln(x)}}$ is positive and decreasing for $x \ge 4$ so we can use the integral test. Therefore we want to consider the integral $\int_4^{\infty} \frac{1}{\ln(x)^{\ln(x)}}dx$. Now we should do a u substitution setting $u = \ln(x)$. Then $du = \frac{1}{x}dx$. Since $u = \ln(x)$ we know that $e^u = x$. Therefore rearranging the equation $du = \frac{1}{x}dx$ we get $e^u du = dx$. Now we get the integral $\int_{\ln(4)} \frac{e^u}{u^u}du = \int_{\ln(4)} (e/u)^u du$. Once $u > e^2$ we see that $(e/u)^u < (1/e)^u = e^{-u}$. The integral $\int_{\ln(4)}^{\infty} e^{-u}du$ converges therefore $\int_{\ln(4)} (e/u)^u du$ converges by comparison. Finally we now know by the integral test that $\sum_{n=4}^{\infty} \frac{1}{\ln(n)^{\ln(n)}}$ converges.

Easier solution:

$$\frac{1}{\ln(n)^{\ln(n)}} = \frac{1}{\left(e^{\ln\ln(n)}\right)^{\ln(n)}} = \frac{1}{e^{\ln\ln(n)\cdot\ln(n)}} = \left(\frac{1}{e^{\ln(n)}}\right)^{\ln\ln(n)} = \frac{1}{n^{\ln\ln(n)}}$$

(p) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3+n^2\cos(n)}$

The best way to approach this series is by using the limit comparison test. n^3 is growing much faster than $n^2\cos(n)$ so n^3 is the dominant term. Therefore we want to take $a_n = \frac{1}{n^3+n^2\cos(n)}$

$= \frac{1}{n^{\ln\ln(n)}}$ is less than $\frac{1}{n^2}$ for big n, so $\sum_{n=4}^{\infty} \frac{1}{\ln(n)^{\ln(n)}}$ converges by Comparison Test since $\sum_{n=4}^{\infty} \frac{1}{n^2}$ converges by p-test ($p=2$).

and $b_n = \frac{1}{n^3}$. Then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^3}{n^3+n^2\cos(n)}$. Since n^3 dominates $n^2\cos(n)$ in the denominator we have $\lim_{n\to\infty}\frac{n^3}{n^3+n^2\cos(n)} = \lim_{n\to\infty}\frac{n^3}{n^3} = 1$. The series $\sum_{n=2}^{\infty}\frac{1}{n^3}$ converges by the p-test with $p=3$ therefore $\sum_{n=2}^{\infty}\frac{1}{n^3+n^2\cos(n)}$ converges by the limit comparison test.

(q) $\displaystyle\sum_{n=0}^{\infty}\frac{e^{n^2}}{n!}$

The presence of factorials and exponentials suggests we should use the ratio test. Therefore we want to consider the limit $\lim_{n\to\infty}\left|\frac{e^{(n+1)^2}n!}{e^{n^2}(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{e^{n^2+2n+1}n!}{e^{n^2}(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{e^{2n+1}}{n+1}\right| = \infty$ since exponentials grow faster than powers. Therefore the series diverges by the ratio test.

(r) $\displaystyle\sum_{n=2}^{\infty}\frac{n}{\ln(n)^3}$

Powers of logarithms always grow slower than any polynomial therefore $\lim_{n\to\infty}\frac{n}{\ln(n)^3} = \infty$ so the series diverges by n^{th} term test ~~since the terms do not go to zero.~~ If you didn't remember this fact you could graph $\frac{x}{\ln(x)^3}$ and see that the graph approaches infinity.

(s) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\sin(1/n)$

$\sin(1/n)$ is decreasing and $\lim_{n\to\infty}\sin(1/n) = 0$. Therefore the above series converges by the alternating series test.

(t) $\displaystyle\sum_{n=0}^{\infty}\frac{9^n}{8^n+10^n}$

There are several ways to approach this series. The fastest way is to use comparison to notice that $\frac{9^n}{8^n+10^n} \leq \frac{9^n}{10^n} = (9/10)^n$. The series $\sum_{n=0}^{\infty}(9/10)^n$ is a convergent geometric series with ratio $9/10$. Therefore $\sum_{n=0}^{\infty}\frac{9^n}{8^n+10^n}$ converges by comparison.

The ratio test and the limit comparison test would also be good tests to use to show that this series converges but they will require more effort. For the limit conmparison test you would want to choose $a_n = \frac{9^n}{8^n+10^n}$ and $b_n = (9/10)^n$.

WARNING: Although factorials dominate exponentials, in the sense that $\lim_{n\to\infty}\frac{n!}{e^n} = \infty$, this example shows that things are different if we use e^{n^2} rather than e^n. Be aware of this possibility.