Math 116 – Team Homework #3, Winter 2023

SOLUTIONS

Some guidelines for your assignment

- You must *read* and *attempt* the problems *before* meeting with your team. Even if you aren't able to obtain all the answers, being prepared during the team meetings helps your group work more efficiently during the meeting.
- Don't be discouraged if you cannot solve most of the problems on your own — this is perfectly normal. This is part of why you are being assigned to work on these assignments as a group; make sure to discuss your questions and ideas with your teammates.
- If your team is having trouble with a particular problem, try utilizing the Math Lab (our math tutoring center - see more details here: `https://lsa.umich.edu/math/undergraduates/course-resources/math-lab.html`) with your teammates to get help.
- Make sure *everyone* is involved and no-one feels excluded during the meetings. If you notice someone is shy, actively encourage them to contribute to the group!
- Ask your teammates to explain their reasoning behind their answers if you don't understand it. Remember that all members of the team are responsible for this assignment, and *everyone* should be on board with what the team turns in.
- Write up your final solutions neatly, and make sure your explanations are clear and complete.
- Consult pages 12-14 of the Student Guide on the course website for more details regarding best practices and team homework roles.

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1. The government has assigned a team to excavate a buried treasure. The team digs and comes across a box, with the top of the box located 25 meters below ground level. The box is a cube, with side length 3 meters. The team lifts the box to ground level as follows.

   A rope, of mass 10 kg per meter, is attached to the box. The box, along with the rope, is lifted via heavy machinery at a constant speed until the top of the box is at ground level. As the box is being lifted, mud plastered to the sides of the box begins falling off at a constant rate of 0.5 kg per second. It takes a total of 2 minutes for the team to lift the top of the box to ground level. Once the box has been lifted, its mass (along with any remaining mud on its sides) is calculated to be 1500 kg.

   (a) Write an expression involving integrals for the total work done, in Joules, in this lifting process. Do not evaluate any integrals. *Be careful to note that only the rope below ground level will contribute to the mass being lifted.*

Solution: Many important quantities for solving this problem are functions of time. Let's explicitly compute these as functions of t with t having units of seconds.

Let $h(t)$ be the height that the box has been lifted after t seconds. Since it is lifted a total of 25 m over the course of 120 s, we have

$$h(t) = \frac{25}{120}t \text{ meters.}$$

Let $m(t)$ be the mass of the muddy box (in kg) after t seconds. We know that $m(120) = 1500$ and that $m'(t) = -0.5$. Therefore

$$m(t) = 1500 - 0.5(t - 120) \text{ kg.}$$

At time t the length of the rope is $25 - \frac{25}{120}t$, so the mass of the rope is:

$$m_{\text{rope}}(t) = (25 - \frac{25}{120}t) \cdot 10 \text{ kg.}$$

Over the course of a small amount of time between t and $t + \Delta t$, the total mass of the system (rope and muddy box) therefore is approximately

$$\begin{aligned} \text{total mass} &= \text{mass of rope} + \text{mass of muddy box} \\ &\approx (25 - \frac{25}{120}t) \cdot 10 + (1500 - 0.5(t - 120)) \text{ kg.} \end{aligned}$$

The amount that the system gets lifted by during this time is approximately

$$\Delta h \approx \frac{25}{120}\Delta t \text{ meters.}$$

Therefore the amount of work done over this time step is approximately

$$\left(\left(25 - \frac{25}{120}t\right) \cdot 10 + \left(1500 - 0.5(t - 120)\right)\right) \cdot 9.8 \cdot \frac{25}{120}\Delta t \quad \text{Joules.}$$

Therefore the total work is

$$\int_0^{120} \left(\left(25 - \frac{25}{120}t\right) \cdot 10 + \left(1500 - 0.5(t - 120)\right)\right) \cdot 9.8 \cdot \frac{25}{120}dt \quad \text{Joules.}$$

The box is transported to the museum. There, after carefully cleaning off all the mud on its sides, the box is opened from the top. The museum desires to transfer the contents of the box to a separate container as it must undergo a series of processes (examining, cleaning, polishing) before the exhibit. The box is full and contains a variety of valuable items, organized in various layers. The mass density of a layer, h meters above the bottom of the box, is given by $\delta(h) = 25(3-h)\frac{\text{kg}}{\text{m}^3}$. The sides, bottom and top of the box are very thin, so you can assume their thickness is 0 m.

(b) Write an expression involving integrals for the total work done, in Joules, to lift the contents of the box to a container placed **2 meters above** the top of the box? Do not evaluate any integrals.

Solution: Let h denote the height in meters from the bottom of the box. The slice of the box at height h has area 9m^2. Therefore the mass (in kg) of a slice between heights h and $h + \Delta h$ is approximately

$$9 \cdot 25(3-h)\Delta h \text{ kg}.$$

This slice needs to get moved up to a height of 5 (because the top of the box has a height of 3 meters, and then the slice needs to get moved an additional 2 meters above the top of the box), namely a distance of $5 - h$ meters. Therefore the total work is

$$\int_0^3 9 \cdot 25(3-h) \cdot (5-h) \cdot 9.8 \, dh \ \text{ Joules.}$$

2. To attract people for their new exhibit, the museum announces a raffle giveaway of a few small items from the treasure box. To prevent blanket entries, the museum devises an online questionnaire involving the following calculus questions. Successfully uploading correct solutions to all of them provides one entry to the raffle.

(a) Does the improper integral $\int_5^\infty xe^{-x}\,dx$ converge or diverge? If it converges, find its **exact value**. If it does not converge, use a **direct computation** of the integral to show its divergence. If your answer does not use **proper notation** and/or does not show your full computation, the questionnaire evaluators reserve the right to disqualify your raffle entry.

Solution:

$$\int_5^\infty xe^{-x}dx = \lim_{b\to+\infty} \int_5^b xe^{-x}dx$$

integration by parts:
$u = x \quad u' = 1$
$v' = e^{-x} \quad v = -e^{-x}$

$$= \lim_{b\to+\infty} \left(-xe^{-x}\Big|_5^b + \int_5^b e^{-x}dx \right)$$

$$= \lim_{b\to+\infty} \left(-\frac{b}{e^b} + 5e^{-5} - e^{-x}\Big|_5^b \right)$$

$$= -\lim_{b\to+\infty} \frac{b}{e^b} + 5e^{-5} - \lim_{b\to+\infty} e^{-b} + e^{-5}$$

$$= -\lim_{b\to+\infty} \frac{b}{e^b} + 5e^{-5} - 0 + e^{-5}$$

$$= -\lim_{b\to+\infty} \frac{1}{e^b} + 6e^{-5} \qquad \text{(by L'Hopital's rule)}$$

$$= 6e^{-5}$$

(b) Does the improper integral $\displaystyle\int_1^\infty \frac{1}{x+\sin x}\,dx$ converge or diverge? **Fully justify** your answer including using proper notation and showing mechanics of any tests you use, but you do not need to evaluate any integrals for this part.

Solution: We note that for $x \geq 1$, we have

$$0 < x + \sin x \leq x + 1 \leq x + x = 2x$$

and therefore

$$\frac{1}{x+\sin x} \geq \frac{1}{2x} > 0.$$

The integral $\displaystyle\int_1^\infty \frac{dx}{2x}$ diverges by the p-test (with $p = 1$). Applying the comparison test, we conclude that the integral $\displaystyle\int_1^\infty \frac{dx}{x+\sin x}$ diverges as well.

May the odds be ever in your favor!