Math 116 — Practice for Exam 1

Generated February 1, 2023

Name: SOLUTIONS

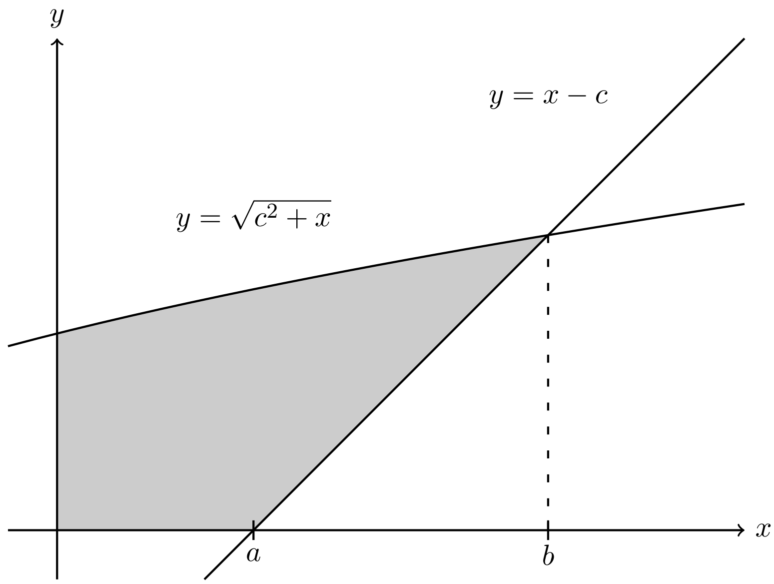
Instructor: ______________________ Section Number: __________

1. This exam has 4 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2016	1	5	graphene	7	
Fall 2017	1	6	rattleback	8	
Fall 2021	1	5	Chicago Bean	11	
Winter 2022	1	7	ship	13	
Total				39	

Recommended time (based on points): 35 minutes

5. [7 points] On his day off, Dr. Durant is experimenting with graphene, a remarkable material that comes in thin sheets. The graphene sample he is currently working with is shaped like the region in the first quadrant shaded below, where $c > 0$ is some positive constant and the units of the axes are mm. Suppose that the mass density of the sample is given by $\delta(x)$ g/mm^2.



a. [3 points] Find a and b. Your answers may include c.

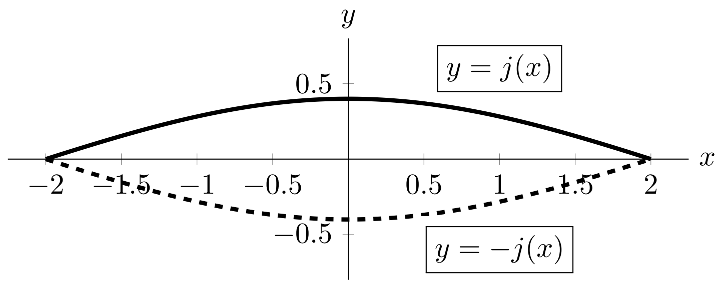
Solution: We have $a = c$ and $b = 2c + 1$.

b. [4 points] Write, but do **not** evaluate, an expression involving integrals that gives the mass of the sample.

Solution: The mass is given by

$$\int_0^c \delta(x)\sqrt{c^2+x}\,dx + \int_c^{2c+1} \delta(x)\left(\sqrt{c^2+x} - (x-c)\right)dx \qquad \text{g.}$$

6. [8 points] A rattleback top is a toy that exhibits interesting physical properties. The toy can be modeled by a solid whose base is the region between the graphs of $j(x)$ and $-j(x)$, shown below. The cross sections perpendicular to the x-axis are semicircles.



The graph of $j(x)$ is solid, the graph of $-j(x)$ is dashed, and the units on both axes are centimeters. Both graphs are bounded between the vertical lines $x = -2$ and $x = 2$.

a. [5 points] Set up, but do **not** evaluate, an expression involving one or more integrals that gives the volume, in cubic centimeters, of the solid rattleback top. Your answer may involve the function name j.

Solution: By slicing perpendicular to the x-axis, a cross-section at horizontal coordinate x will have height $j(x) - (-j(x)) = 2j(x)$. This is the diameter of our semicircular cross-section, so the radius of the cross section is then half the diameter, or $\frac{2j(x)}{2} = j(x)$. A semicircle of radius r has area $\frac{\pi}{2}r^2$, so if each slice has thickness Δx, then the volume of such a slice can be approximated by

$$V_{slice} \approx \frac{\pi}{2}(j(x))^2 \Delta x.$$

We compute the total volume by summing over all slices and taking the limit as the thickness of the slices approaches 0. The total volume is then

$$V_{total} = \int_{-2}^{2} \frac{\pi}{2}(j(x))^2 \, dx \ \text{ cm}^3.$$

b. [3 points] In order to make the rattleback top spin like a top, it is made out of plastic that has a mass density given by the function $\delta(x)$ grams per cubic centimeter, where x is the x-coordinate in the diagram above. Set up, but do **not** evaluate, an expression involving one or more integrals that gives the mass, in grams, of the rattleback top. Your answer may involve the function names j and/or δ.

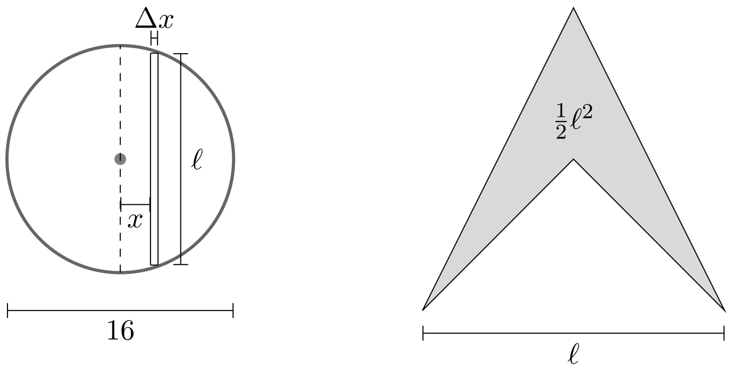
Solution: For small Δx, the mass of a slice as described in part **a** is approximately

$$m_{slice} \approx \frac{\pi}{2}(j(x))^2 \delta(x) \Delta x.$$

Hence the total mass is

$$m_{total} = \int_{-2}^{2} \frac{\pi}{2}(j(x))^2 \delta(x) \, dx \ \text{ grams.}$$

5. [11 points] Tony's climbing gym wants to put in a climbing structure based off of the Chicago Bean. However, they want to make it more angular. The base of the structure will be in the shape of a circle with an 8 meter radius. The cross-sections perpendicular to the circle lying above a slice of the circle of length ℓ meters (as shown below) have area $\frac{1}{2}\ell^2$ square meters and are pictured below. The density of the material used to build the structure is not constant and has density dependent on its horizontal distance x from the vertical diameter through the circle. The density in kg/m^3 is given by $\delta(x) = 1000\sqrt{1+x^2}$.



a. [2 points] Write an expression that gives the quantity ℓ in terms of x.

Solution: Using x and ℓ as defined in the figure above, we have $x^2 + \left(\frac{\ell}{2}\right) = 8^2$. Solving this, since $\ell \geq 0$,

$$\ell = 2\sqrt{64 - x^2}.$$

b. [3 points] Write an expression that gives the approximate volume, in cubic meters, of a slice of the structure a horizontal distance x meters away from the diameter of the circle with thickness Δx. Your expression should not involve any integrals.

Solution: The approximate volume of a slice of the structure x meters away from the vertical diameter of the circle, using (a) is

$$\frac{1}{2}\ell^2\Delta x = 2(64 - x^2)\Delta x.$$

c. [3 points] Using your expression from (b) to write an expression involving integrals which gives the total volume of the structure in cubic meters. Do not evaluate any integrals.

Solution: The volume is the integral of the volume of the slice in (b) as $\Delta x \to 0$, from $x = -8$ to $x = 8$ (or alternatively twice the integral from $x = 0$ to $x = 8$):

$$2\int_0^8 2(64 - x^2)dx = 4\int_0^8 64 - x^2 dx.$$

d. [3 points] Write an expression involving integrals which gives the total mass of the structure in kg. Your answer may contain $\delta(x)$. Do not evaluate any integrals.

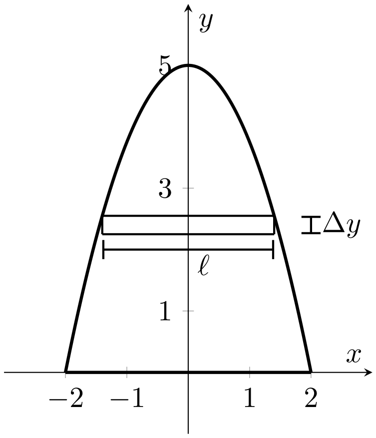
Solution: Using our answer from (b) and the density of the slice a horizontal distance x meters away from the vertical diameter of the circle, the approximate mass of a slice of the structure is

$$2\delta(x)(64 - x^2)\Delta x.$$

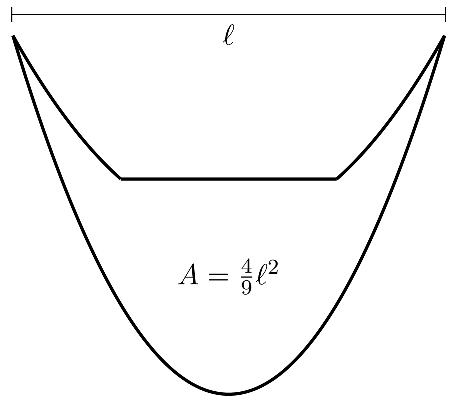
The mass is the integral of the mass of the slice as $\Delta x \to 0$ from $x = -8$ to $x = 8$ (or alternatively twice the integral from $x = 0$ to $x = 8$ since $\delta(x)$ is even):

$$4\int_0^8 \delta(x)(64 - x^2)dx.$$

7. [13 points] Brad and Shawna are shipwrecked on an island and are building a new ship out of various materials. The ship has a base given by the region enclosed in the figure on the left, with cross-sections perpendicular to the y-axis given by the figure on the right. The base is the region bounded by $y = \frac{-5}{4}(x^2 - 4)$ and $y = 0$. The cross-sections have area given by $\frac{4}{9}\ell^2$ where ℓ is the length of the slice of the base directly below the cross-section. A sample slice of the base of thickness Δy is shown in graph on the left, and all distances are given in meters.



Base of Ship



Cross-section of Ship

a. [3 points] Write an expression for the length, ℓ, of a slice y meters from the x-axis. Give units.

Solution:

$$\ell = 2\sqrt{\frac{-4}{5}y + 4} \text{ m.}$$

b. [3 points] Write an expression for the volume of materials needed to construct a cross-sectional slice of the ship y meters from the x-axis with thickness Δy meters. The letter ℓ should not appear in your final answer. Give units.

Solution:

$$\frac{4}{9}\left(2\sqrt{\frac{-4}{5}y + 4}\right)^2 \Delta y \text{ m}^3 = \frac{16}{9}\left(\frac{-4}{5}y + 4\right)\Delta y \text{ m}^3$$

c. [3 points] The density of the materials used to make the ship varies. The materials used in the cross section y meters from the x-axis is given by $\delta(y) = (2y + 5)$ kg/m^3. What is the mass of a cross sectional slice y meters from the x-axis with thickness Δy meters? Give units.

Solution:

$$\frac{16}{9}(2y + 5)\left(\frac{-4}{5}y + 4\right)\Delta y \text{ kg}$$

d. [4 points] Write an integral that gives the total mass of the new boat in kg. Do not evaluate your integral.

Solution:

$$\frac{16}{9}\int_0^5 (2y+5)\left(\frac{-4}{5}y+4\right)dy \text{ kg}$$