Math 116 — Practice for Exam 2

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Name: SOLUTIONS

Instructor: ____________________ Section Number: __________

1. This exam has 3 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2022	2	5	data storage	12	
Fall 2021	2	3	sheep	14	
Winter 2021	2	2	log pile	12	
Total				38	

Recommended time (based on points): 34 minutes

5. [12 points] A tech startup is growing quickly, and the company needs to understand its customers data-storage needs to properly scale its infrastructure. Over the course of each month, the users each store 5 gigabytes of new data. Additionally, because users are conscious of their digital footprint, at the beginning of each month, each user deletes 20% of all data they had stored in previous months.

a. [4 points] Let D_n be the amount of data stored per user at the end of the n^{th} month. If $D_1 = 5$, write expressions for D_2 and D_3. The letter D should not appear in your final answers.

$D_2 =$ $5 + 5\,(.8)$

$D_3 =$ $5 + 5\,(.8) + 5\,(.8)^2$

b. [4 points] Find a closed form expression for D_n. This means your answer should be a function of n, should not contain Σ, and should not be recursive.

$D_n =$ $\dfrac{5(1-(.8)^n)}{1-(.8)}$

c. [4 points] What is the long-term expected data storage of a user in gigabytes?

Answer $=$ $\dfrac{5}{1-.8} = 25$

3. [14 points] Molly has recently become a sheep herder. She rotates her sheep through various fields so that the sheep have a varied diet and the fields have a chance to grow. Every Monday, the sheep visit the same field. Before the sheep graze for the first time in this field, its grass is 20 centimeters tall. Molly's sheep are picky and only eat the top 40% of the length of grass in this field every Monday. Over the course of the week, before the next Monday, the grass grows 3 centimeters. Let G_i represent the height in centimeters of the grass right before the sheep graze on it for the ith time. Note that $G_1 = 20$.

a. [5 points] Find expressions for each of G_2, G_3, and G_4. You do not need to evaluate your expressions.

Solution:

$$\begin{aligned} G_2 &= (0.6)G_1 + 3 \\ &= (0.6)(20) + 3 \\ G_3 &= (0.6)G_2 + 3 \\ &= (0.6)^2(20) + (0.6)(3) + 3 \\ G_4 &= (0.6)G_3 + 3 \\ &= (0.6)^3(20) + (0.6)^2(3) + (0.6)(3) + 3 \end{aligned}$$

b. [5 points] Find a general **closed-form** expression for G_n, defined for $n = 2, 3, 4 \ldots$

Solution:

$$\begin{aligned} G_n &= (0.6)^{n-1}(20) + \sum_{i=0}^{n-2} 3(0.6)^i \\ &= (0.6)^{n-1}(20) + \frac{3(1-(0.6)^{n-1})}{1-0.6} \end{aligned}$$

c. [4 points] In order for the field to meet sheep grazing standards, the height of the grass must be at least 5 cm when the sheep begin grazing. Molly thinks she will be able to stay on her field forever. Help her determine whether she can stay by either showing that the grass will eventually be less than 5 cm in height, or showing that the grass will be at least 5 cm each time before the sheep graze.

Solution:

$$\lim_{n\to\infty} G_n = \frac{3}{1-0.6} = 7.5.$$

Also note that G_n is a decreasing sequence. So, the grass is always taller than 5 cm. when the sheep begin grazing.

2. [12 points] In order to build a settlement on the island, intruders start cutting down trees at the forest, cutting the trees into logs, and putting the logs in a pile. Let A_n be the number of logs they have in the pile at noon on the n-th day. The intruders have 100 logs in the pile at noon on the first day (so $A_1 = 100$). Every day (between noon on one day and noon on the next day), the building team uses 10% of the logs in the pile, while the log-cutting team adds 20 logs to the pile immediately before noon.

a. [4 points] Find A_2 and A_3. **You do not need to simplify your answers.**

Solution:

$$A_2 = 100 \cdot 0.9 + 20$$

$$A_3 = (100 \cdot 0.9 + 20) \cdot 0.9 + 20 = 100 \cdot 0.9^2 + 20 + 20 \cdot 0.9$$

b. [5 points] Find a **closed form** expression for A_n. Closed form means your answer should not include ellipses or sigma notation, and should NOT be recursive. **You do not need to simplify your closed form answer.**

Solution: From observing the pattern from A_1, A_2 and A_3, we have

$$\begin{aligned} A_n &= 100 \cdot 0.9^{n-1} + (20 + 20 \cdot 0.9 + 20 \cdot 0.9^2 + \cdots + 20 \cdot 0.9^{n-2}) \\ &= 100 \cdot 0.9^{n-1} + 20 \cdot \frac{1 - 0.9^{n-1}}{1 - 0.9}. \end{aligned}$$

Note that the term $100 \cdot 0.9^{n-1}$ is not part of the geometric series. There are $n-1$ terms in the geometric series, so the exponent in the closed form is $n-1$.

c. [3 points] How many logs will the intruders have in the pile in the long run?

Solution:

$$\lim_{n\to\infty} 100 \cdot 0.9^{n-1} + 20 \cdot \frac{1 - 0.9^{n-1}}{1 - 0.9} = 0 + 20 \cdot \frac{1}{1 - 0.9} = 20 \cdot 10 = 200.$$