WRITING INEQUALITIES (ABRIDGED)

1. Exercises

(1) Write down inequalities to discuss the convergence/divergence of the following integrals.

a) $\displaystyle\int_3^\infty \frac{\arctan(x)}{e^{5x}}\,dx$

b) $\displaystyle\int_1^\infty \frac{1}{xe^x}\,dx$

c) $\displaystyle\int_0^\infty x^2e^{-x^4}\,dx$

d) $\displaystyle\int_0^1 \frac{1}{x^3+x^{1/2}}\,dx$

e) $\displaystyle\int_0^{\pi/2} \frac{\sin(x)}{x^{5/2}}\,dx$

2. Solutions

The solutions below are written as follows. The intuition behind the argument (and the intuition for how one might approach each problem) is written in magenta. The rigorous solution is then written in blue.

(1) (a) Informally, we know that while $\arctan(x)$ is increasing, it is still bounded above by $\pi/2$. Thus, the integrand $\frac{\arctan(x)}{e^{5x}}$ should have the same behavior as e^{-5x} so we expect convergence.
This means that we need a convergent upper bound. In this case, we can just use the natural restriction of the range for $\arctan(x)$.

We have that, for $x \geqslant 3$,

$$\arctan(x) \leqslant \frac{\pi}{2}$$

so that

$$\frac{\arctan(x)}{e^{5x}} \leqslant \frac{\pi}{2e^{5x}}.$$

By the exponential decay test, we know that $\displaystyle\int_3^\infty \frac{\pi}{2e^{5x}}\,dx$ converges.
Therefore, by the comparison test

$$\int_3^\infty \frac{\arctan(x)}{e^{5x}}\,dx$$

also converges.

(b) Informally, the denominator is growing faster than a growing exponential, because of the x term. Thus, we should expect the integral to converge.
To produce an upper bound that is convergent, note that we cannot drop the e^x term because then the denominator does not grow fast enough. On the other hand, the x-term is not necessary for the denominator to grow fast enough. So we can try dropping it.
We need an inequality of the form

$$\frac{1}{xe^x} \leqslant \frac{C}{e^x}$$

for $x \geqslant x_0$. (We need to work out C and x_0 to make this statement precise.)
But note that our desired inequality can be rearranged as

$$\frac{1}{xe^x} \leqslant \frac{C}{e^x}$$

or

$$e^x \leqslant Cxe^x.$$

Cancelling the e^x terms on both sides gives us

$$1 \leqslant Cx.$$

We see that if $x \geqslant 1$, we can simply take $C = 1$. Taking any constant $C \geqslant 1$ would also work. Thus, we see that

$$\frac{1}{xe^x} \leqslant \frac{1}{e^x}$$

for all $x \geqslant 1$.

Now, by exponential decay test, we have that $\int_1^\infty e^{-x}\,dx$ converges.

By the comparison test, we have that $\int_1^\infty \frac{1}{xe^x}\,dx$ converges.

(c) Informally, in our integrand $\frac{x^2}{e^{x^4}}$, it is true that the numerator is growing and so cancelling out some of the growth in the denominator, but still, the exponential growth is going to win out. This means that we should expect the integral to converge.

To prove this using the comparison test, we'd need a convergent upper bound. If you noticed that the derivative of x^4, which is $4x^3$, is almost (but not quite!) there in the integrand, you can use the comparison technique to get to a comparison function that we can integrate using techniques from this class.

Note that $x^2e^{-x^4} \leqslant x^3e^{-x^4}$ for $x \geqslant 1$. (This is because, when $x \geqslant 1$, we have $x^2 \leqslant x^3$.)

Now, the comparison function we have arrived at is not a standard comparison function but we can integrate it directly.

$$\int_1^\infty x^3e^{-x^4}\,dx = \lim_{B\to\infty}\int_1^B x^3e^{-x^4}\,dx$$

Carrying out the indefinite integral fully before proceeding,

$$\begin{aligned}\int x^3e^{-x^4}\,dx \quad &\overset{\substack{u=x^4\\ du=4x^3dx}}{=} \int e^{-u}\frac{du}{4}\\ &= -\frac{1}{4}e^{-u}\\ &= -\frac{1}{4}e^{-x^4}.\end{aligned}$$

Now, our improper integral becomes:

$$\begin{aligned}\int_1^\infty x^3e^{-x^4}\,dx &= \lim_{B\to\infty} -\frac{1}{4}e^{-x^4}\Big|_1^B\\ &= \lim_{B\to\infty}\left(-\frac{1}{4}e^{-B^4}+\frac{e^{-1}}{4}\right)\end{aligned}$$

Moving the constants out of the limit

$$= -\frac{1}{4}\lim_{B\to\infty}(e^{-B^4}) + \frac{e^{-1}}{4}$$

Since we have that $-B^4 \to -\infty$ as $B \to \infty$, we have that $e^{-B^4} \to 0$. Thus

$$\int_1^\infty x^3e^{-x^4}\,dx = \frac{1}{4e}.$$

In particular, the improper integral of our comparison function converges. Therefore, by comparison test,

$$\int_1^\infty x^2e^{-x^4}\,dx$$

also converges (and has value less than $e/4$).
Now, $\int_0^1 x^2 e^{-x^4}\,dx$ is a proper integral (and hence finite), it follows that $\int_0^\infty x^2 e^{-x^4}\,dx$ converges.

(d) Informally, the larger term in the denominator is $x^{1/2}$ as $x \to 0$. This a very important point—because our integral is improper near 0. Once you get that, you have that the integrand should behave like $\dfrac{1}{x^{1/2}}$ near $x = 0$. Therefore, by the second integral p-test with $p = 1/2$, the integral converges.
So, we need a convergent upper bound.
By dropping positive terms in the denominator, we know that the fraction becomes bigger; this gives the inequality that we want. Namely, we have that

$$\frac{1}{x^3 + x^{1/2}} \leqslant \frac{1}{x^{1/2}}$$

for $0 < x \leq 1$.
By the second integral p-test with $p = 1/2$, the integral $\displaystyle\int_0^1 \frac{1}{x^{1/2}}\,dx$ converges.
Therefore by the comparison test, the integral $\displaystyle\int_0^1 \frac{1}{x^3 + x^{1/2}}\,dx$ converges.

(e) First, the integral is improper because of the integrand's behavior near 0. Recall that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

(Use L'Hopital's rule to check this if you don't know this limit.) We interpret this as saying that $\sin(x) \approx x$ near $x = 0$. Therefore near $x = 0$ the integrand should behave like $\dfrac{x}{x^{5/2}} = \dfrac{1}{x^{3/2}}$.
By the second integral p-test with $p = 3/2$, we expect divergence. Thus, we need a lower bound.
We can use a graphical approach here. Note that $\sin(x)$ is concave down on $[0, \pi/2]$ so that secant line between the points $(0, 0)$ and $(\pi/2, 1)$ of the graph of $\sin(x)$ lies below the graph of $\sin(x)$.
This gives us that $\sin(x) \geqslant \frac{2}{\pi}x$.
Therefore, we get

$$\frac{\sin(x)}{x^{5/2}} \geqslant \frac{2x}{\pi x^{5/2}} = \frac{2}{\pi x^{3/2}}$$

for x in the interval $(0, \pi/2]$.
Now, by the second integral p-test with $p = 3/2$, the integral $\displaystyle\int_0^{\pi/2} \frac{2}{\pi x^{3/2}}\,dx$ diverges.
Therefore by comparison test, the integral $\displaystyle\int_0^{\pi/2} \frac{\sin(x)}{x^{5/2}}\,dx$ diverges.