Math 494, Homework 1: due Jan 28

(1) If R is a (commutative) ring and $f(x) \in R[x]$, then an n-cycle of f in R is defined to be an n-tuple $(c_1, c_2, \dots, c_n)$ of pairwise distinct elements of R such that $f(c_i) = c_{i+1}$ for $1 \le i \le n$ (where we define $c_{n+1} := c_1$). Let $\mathcal{CYC}(R)$ be the set of cycle lengths of polynomials over R (so $\mathcal{CYC}(R) = \{n\colon$ there exists $f \in R[x]$ with an n-cycle in $R\}$). Show that $\mathcal{CYC}(\mathbf{Z}) = \{1, 2\}$.

(2) Show that if R is an integral domain (i.e., a commutative ring in which $ab = 0$ implies $0 \in \{a, b\}$) and p is a prime number, then $p \in \mathcal{CYC}(R)$ if and only if there exist units $u_1, u_2, \dots, u_{p-1} \in R^*$ such that $u_i - u_j \in R^*$ whenever $0 < i < j < p$. Deduce that if p, q are prime numbers with $p < q$, and $q \in \mathcal{CYC}(R)$, then $p \in \mathcal{CYC}(R)$.

(3) Let E be the set of entire functions on the complex plane, which by definition is the set of functions defined by a single power series $\sum_{n=0}^{\infty} a_n x^n$ in $\mathbf{C}[[x]]$ which converges whenever $x \in \mathbf{C}$. It is easy to verify that E^* consists of all elements of E which do not take value 0 anywhere on $\mathbf{C}$. It can also be shown (via logarithms) that $E^* = \{e^f \colon f \in E\}$. In 1887(!!), Borel showed that if $f_1, \dots, f_n \in E^*$ satisfy $f_1 + \dots + f_n = 0$, but no nonempty proper subset of the f_i's sums to zero, then all the f_i's are constant multiples of one another (i.e., $f_i/f_j \in \mathbf{C}^*$ for all i, j). Assuming this, deduce that if $f, g \in E^*$ satisfy $H(f, g) = 0$ for some nonzero $H(x, y) \in \mathbb{C}[x, y]$, then $f^m = c \cdot g^n$ for some $c \in \mathbb{C}^*$ and some integers m, n which are not both zero.

(4) Problems 1.3, 1.5, 1.6, 1.7, 1.9 from chapter 11 of Artin.