Math 494, Homework 10: due Thursday April 8

(1) Let L/K and M/K be finite-degree separable field extensions, and suppose that L/K and M/K are *minimal* in the sense that there are no fields strictly between L and K, and also there are no fields strictly between M and K. Show that if $[LM : K] < [L : K] \cdot [M : K]$ then the Galois closure of L/K equals the Galois closure of M/K.

(1.5) *(This problem is optional, and is only for extra credit.)*
Let L/K and M/K be finite-degree separable field extensions. Show that if $[LM : M] < [L : K]$ then there exist fields L_1 and M_1 such that all of these hold:
- $K \subsetneq L_1 \subseteq L$ and $K \subsetneq M_1 \subseteq M$
- $[L_1 M_1 : K] < [L_1 : K] \cdot [M_1 : K]$
- the Galois closure of L_1/K equals the Galois closure of M_1/K.

(2) Let L/K and M/K be separable extensions which both have degree n, where $n \neq 6$. Let N be the normal closure of L/K. Show that if $\text{Gal}(N/K) \cong S_n$ then $[LM : M] \in \{1, n-1, n\}$.
Hint: You may use the fact that if $n \neq 6$ then any two index-n subgroups H, J of S_n are conjugate to one another, and there are no groups strictly between H and S_n. I have posted proofs of these facts in piazza.

(3) Let p be a prime which is not in $\{11, 23\}$ and which cannot be written as $(q^d-1)/(q-1)$ with $d \geq 2$ and q a prime power. Let $f(X)$ be an irreducible degree-p polynomial in $K[X]$, for some field K, and let L/K be a separable extension of degree a power of p. Show that every irreducible factor of $f(X)$ in $L[X]$ has degree either p or a divisor of $p - 1$.
Hint: use Theorem 5.5 from the notes, which is a consequence of the classification of finite simple groups. I can't imagine how one could solve this problem without relying on this classification. I will post on Piazza some examples for the excluded primes p, in which different degrees occur.

(4) Let N be a field and let G be a finite group of automorphisms of N. Writing $N^G := \{n \in N : g(n) = n \ \forall g \in G\}$, show that N/N^G is Galois with Galois group G.