Math 494, Homework 11: due Thursday April 15

(1) Let K be a field, and let $f(X) \in K[X]$ be a degree-n polynomial which has n distinct roots $\alpha_1, \dots, \alpha_n$ in its splitting field. Show that $\mathrm{Gal}(f, K)$, when viewed as a group of permutations of $\{\alpha_1, \dots, \alpha_n\}$, has orbit sizes equal to the degrees of the irreducible factors of $f(X)$ in $K[X]$. In particular, $f(X)$ is irreducible in $K[X]$ if and only if $\mathrm{Gal}(f, K)$ is a transitive subgroup of S_n. In case $K = \mathbb{F}_q$, show that $\mathrm{Gal}(f, \mathbb{F}_q)$ is generated by an element of S_n whose cycle lengths (including 1-cycles) equal the degrees of the irreducible factors of $f(X)$ in $\mathbb{F}_q[X]$.

(2) Let K be a field whose characteristic is not 2, and let $f(X) \in K[X]$ have degree $n > 0$ and have n distinct roots $\alpha_1, \dots, \alpha_n$ in its splitting field. Show that $\mathrm{Gal}(f, K)$ is a subgroup of A_n if and only if $\prod_{i<j}(\alpha_i - \alpha_j)^2$ is a square in K. Express this quantity $\prod_{i<j}(\alpha_i - \alpha_j)^2$ in terms of the coefficients of $f(X)$ in case $\deg(f) = 2$, and in case $f(X) = X^3 + aX + b$.

(3) Let $x_1, \dots, x_n$ be independent transcendentals over a field K, and write $\prod_{i=1}^n (X - x_i) = X^n - e_1 X^{n-1} + e_2 X^{n-2} - \cdots + (-1)^n e_n$. Show that $K(x_1, \dots, x_n)/K(e_1, \dots, e_n)$ is Galois with Galois group S_n. In case the characteristic of K is not 2, name an element $\Delta \in K(x_1, \dots, x_n)$ so that $K(x_1, \dots, x_n)^{A_n} = K(e_1, \dots, e_n, \Delta)$.