Math 494, Homework 3: due Feb 11

(1) An *algebraic variety* in $\mathbb{C}^n$ is the set of common zeroes in $\mathbb{C}^n$ of some set of polynomials in $\mathbb{C}[x_1, ..., x_n]$. Let Σ_n be the set of all algebraic varieties in $\mathbb{C}^n$. Show that
   (1) Σ_n contains both $\emptyset$ and $\mathbb{C}^n$
   (2) If $A, B \in \Sigma_n$ then $A \cup B \in \Sigma_n$
   (3) If $(A_\lambda)_{\lambda\in\Lambda}$ is any family of elements of Σ_n then $\cap_{\lambda\in\Lambda} A_\lambda$ is in Σ_n.

(2) Let R be a (commutative) ring and let S be the set of all maximal ideals of R. For each subset T of R, let $V(T)$ be the set of all maximal ideals of R which contain T. Show that
   (1) If I is the ideal generated by T then $V(T) = V(I)$.
   (2) $V(0) = S$ and $V(1) = \emptyset$.
   (3) If $(T_\lambda)_{\lambda\in\Lambda}$ is any family of subsets of R then
   $$V\big(\cup_{\lambda\in\Lambda} T_\lambda\big) = \cap_{\lambda\in\Lambda} V(T_\lambda).$$
   (4) For any ideals I, J of R we have $V(I \cap J) = V(IJ) = V(I) \cup V(J)$.

   Remark: For those of you who have seen the notion of an abstract topological space, note that problem (1) says that the algebraic varieties in $\mathbb{C}^n$ satisfy the axioms for being the closed sets in a topological space, which gives a novel topological structure to $\mathbb{C}^n$. Likewise, problem (2) says that the sets $V(T)$ are the closed sets in a topological space, which yields a topological structure on the set of maximal ideals of R.

(3) Let X be an algebraic variety in $\mathbb{C}^n$, and let $I(X)$ be the set of all polynomials $f(x_1, \dots, x_n) \in \mathbb{C}[x_1, ..., x_n]$ which vanish at all elements of X. Show that $I(X)$ is an ideal. The quotient ring $C(X) := \mathbb{C}[x_1, ..., x_n]/I(X)$ is called the "coordinate ring" of X. Let Y be an algebraic variety in $\mathbb{C}^m$ with coordinate ring $C(Y)$. A *polynomial mapping* is a function $\phi\colon X \to Y$ which is given by an m-tuple of n-variable polynomials. Exhibit a bijection between the set of such polynomial mappings $X \to Y$ and the set of ring homomorphisms $C(Y) \to C(X)$ which act as the identity on $\mathbb{C}$.

(3) Problems 4.3, 7.1, 8.1, 8.2 from chapter 11 of Artin.