Math 494, Homework 6: due Mar 11 (after the midterm)

(1) Let K be a field, and let x be transcendental over K. If $f(X) \in K(X)$ has degree $n > 0$ then show that the field extension $K(x)/K(f(x))$ has degree n.

(2) (a) Let L and M be subfields of a field N, and let F be a subfield of $L \cap M$. Show that if the degrees $[L : F]$ and $[M : F]$ are integers which are relatively prime to one another, then $[LM : F] = [L : F] \cdot [M : F]$.
(b) Let K be a field, and let $f(X)$ and $g(X)$ be nonconstant polynomials in $K[X]$ whose degrees are relatively prime to one another. Show that $f(X) - g(Y)$ is irreducible in $K[X, Y]$.

(3) Problems 2.2, 2.3, 3.5, 3.8, 3.9, 4.2 from chapter 15 of Artin.

(4) If x is transcendental over the field K then show that every field L such that $K \subsetneqq L \subseteq K(x)$ has the form $L = K(f(x))$ for some nonconstant $f(X) \in K(X)$.
(Note: this problem is hard, and a complete solution will earn massive extra credit. I will posted hints for this problem on Piazza.)