Math 676, Homework 11: due Nov 18

(1) Let A be a principal ideal domain with field of fractions K, let B be a Dedekind domain containing A such that B is a finitely-generated A-module, and assume that the field of fractions L of B is a finite extension of K. Show that B is a free A-module of rank $[L:K]$, and also that for any nonzero prime ideal $\mathfrak{p}$ of A the set $B/\mathfrak{p}B$ has dimension n as a vector space over $A/\mathfrak{p}$.

(2) Let $A_1 \subset A_2 \subset A_3$ be Dedekind domains, let $\mathfrak{p}_3$ be a nonzero prime ideal of A_3, and put $\mathfrak{p}_i := A_i \cap \mathfrak{p}_3$. Show that $e(\mathfrak{p}_3/\mathfrak{p}_1) = e(\mathfrak{p}_3/\mathfrak{p}_2) \cdot e(\mathfrak{p}_2/\mathfrak{p}_1)$ and $f(\mathfrak{p}_3/\mathfrak{p}_1) = f(\mathfrak{p}_3/\mathfrak{p}_2) \cdot f(\mathfrak{p}_2/\mathfrak{p}_1)$.

(3) Let L/K be a Galois extension of number fields, let $\mathfrak{q}$ be a nonzero prime ideal of $\mathcal{O}_L$, and let $\mathfrak{p} := \mathfrak{q} \cap \mathcal{O}_K$. Show that $L^{D_{\mathfrak{q}/\mathfrak{p}}}$ is the largest field L' between K and L for which the prime $\mathfrak{p}' := \mathfrak{q} \cap \mathcal{O}_{L'}$ satisfies $e(\mathfrak{p}'/\mathfrak{p}) = 1 = f(\mathfrak{p}'/\mathfrak{p})$. Give a similar characterization of $L^{I_{\mathfrak{q}/\mathfrak{p}}}$.

(4) Let L_1 and L_2 be finite extensions of a number field K, and let $\mathfrak{p}$ be a nonzero prime ideal of $\mathcal{O}_K$. Show that $\mathfrak{p}$ is unramified in both L_1 and L_2 if and only if $\mathfrak{p}$ is unramified in the compositum L_1L_2. Also show that $\mathfrak{p}$ splits completely in both L_1 and L_2 if and only if $\mathfrak{p}$ splits completely in the compositum L_1L_2.

(5) Find a prime number p and quadratic extensions K and L of $\mathbb{Q}$ demonstrating each of the following:
   (a) p can be totally ramified in both K and L without being totally ramified in KL.
   (b) K and L can each contain unique primes lying over p while KL does not.
   (c) p can be inert in K and L without being inert in KL.
   (d) The residue degrees of p in K and L can be 1 without being 1 in KL.

(6) Let L/K be an extension of number fields, let M be the Galois closure of L/K, and let $G := \mathrm{Gal}(M/K)$ and $H := \mathrm{Gal}(M/L)$ be the Galois groups. For each nonzero prime ideal P of $\mathcal{O}_K$, let R be a prime ideal of $\mathcal{O}_M$ lying over P, and let D and I be the decomposition and inertia groups of R/P. Show that there is a bijection between the set of prime ideals of $\mathcal{O}_L$ lying over P and the set of D-orbits on the set of left cosets G/H, where the orbit corresponding to an ideal Q is the union of $f(Q/P)$ disjoint I-orbits, each of which has length $e(Q/P)$.