Math 676, Homework 12: due Dec 2

(1) For each prime $p > 2$, show that $16 = c^8$ for some $c \in \mathbb{Q}_p$.

(2) Show that both $(x^2-2)(x^2-17)(x^2-34)$ and $(x^3-37)(x^2+3)$ have roots in $\mathbb{Q}_p$ for every p, but have no roots in $\mathbb{Q}$.
Massive extra credit: Show that for each prime p the equation $3x^4+4y^4-19z^4 = 0$ has solutions with $x, y, z \in \mathbb{Q}_p$ which are not all zero, but no such solutions with $x, y, z \in \mathbb{Q}$.

(3) Let K be a field which is complete with respect to a non-archimedean absolute value $|\cdot|$, let $a_0, a_1, \dots$ be a sequence of elements of K, and define

$$R := \frac{1}{\limsup_n |a_n|^{1/n}}$$

which is an element of $[0, +\infty]$. Show that $D := \{x \in K \colon \sum_{n=0}^{\infty} a_n x^n \text{ converges}\}$ satisfies:
(1) If $R = 0$ then $D = \{0\}$.
(2) If $R = \infty$ then $D = K$.
(3) If $0 < R < \infty$ and $\lim_n |a_n| R^n = 0$ then $D = \{x \in K \colon |x| \le R\}$.
(4) If $0 < R < \infty$ and $|a_n| R^n \not\to 0$ then $D = \{x \in K \colon |x| < R\}$.

(4) If p is an odd prime, $t \in \mathbb{Z}_p$, and $x \in p\mathbb{Z}_p$, show that the binomial series

$$G(t, x) := \sum_{n=0}^{\infty} \binom{t}{n} x^n$$

converges. If $t = u/v$ with $u, v \in \mathbb{Z}$ and $v > 0$ and $p \nmid v$, then show that $G(\frac{u}{v}, x)^v = (1+x)^u$. Show in particular that if $p = 7$, $t = 1/2$ and $x = 7/9$ then the series converges to $4/3$ in $\mathbb{R}$ and to a 7-adic number $\alpha \neq 4/3$ in $\mathbb{Q}_7$.

(5) (a) Determine the set of elements in $\mathbb{Q}_p$ for which the power series

$$\log_p(x) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$$

converges.
(b) Determine the set of elements in $\mathbb{Q}_p$ for which the power series

$$\exp_p(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges.
(c) If $a \in \mathbb{Q}_p$ is small enough, show that $\exp_p(\log_p(a)) = a$. How close is "close enough"?