Math 676, Homework 2: due Sep 16

(1) Show that the following conditions on a ring R are equivalent:
   (i) Every ideal of R is finitely generated (i.e., R is Noetherian).
   (ii) If $I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$ is an increasing chain of ideals in R, then there exists N such that if $n \geq N$ then $I_n = I_N$.
   (iii) Every nonempty set Σ of ideals of R contains a maximal element in the sense of inclusion; in other words, Σ contains an ideal I which is not properly contained in any other ideal in Σ.

(2) Let R be an integral domain with field of fractions K. Define a *fractional ideal* of R to be an R-submodule J of K such that $\alpha J \subseteq R$ for some nonzero $\alpha \in R$.
   (a) Show that a fractional ideal of R is the same thing as a subset of K of the form $I/\alpha := \{i/\alpha \colon i \in I\}$ with I being an ideal of R and α being a nonzero element of R.
   (b) By definition, the product of two R-submodules J_1, J_2 of K to be the set of all R-linear combinations of elements of the form $j_1 j_2$ with $j_i \in J_i$. Show that the product of two fractional ideals of R is again a fractional ideal.
   (c) Show that if R is noetherian then fractional ideals of R are the same thing as finitely generated R-submodules of K.