Math 676, Homework 3: due Sep 23

(1) Can every proper ideal of $\mathbb{Z}[\sqrt{-3}]$ be written as a product of prime ideals? Are there proper ideals of $\mathbb{Z}[\sqrt{-3}]$ which can be written in more than one way as a product of prime ideals?

(2) Let $x_1, \dots, x_n$ be a basis for a free $\mathbb{Z}$-module G, and let $y_1, \dots, y_n$ be a basis (as a $\mathbb{Z}$-module) for a sub-$\mathbb{Z}$ module H of G with rank n. Write $y_j = \sum_{i=1}^n a_{ij} x_i$ with $a_{ij} \in \mathbb{Z}$, and let A be the n-by-n matrix with ij entry beig a_{ij}. Show that the index $[G : H]$ equals $|\det(A)|$.

(3) Let I and J be proper ideals of a Dedekind domain R. Show:
   (a) if $I = P_1 P_2 \dots P_k$ with each P_i being a prime ideal, then $I^{-1} = P_1^{-1} P_2^{-1} \dots P_k {-1}$
   (b) $II^{-1} = R$
   (c) $I \supseteq J$ if and only if I divides J, in the sense that $IM = J$ for some ideal M of R.