Math 676, Homework 4: due Oct 9

(1) Assuming that the Picard group of $\mathbb{Z}[\sqrt{-19}]$ has order 3, find all integer solutions of $x^2 + 19 = y^5$. Here the *Picard group* of an integral domain R is the quotient of the group of invertible fractional ideals of R by the subgroup of principal fractional ideals, where a fractional ideal I is called *invertible* if there exists a fractional ideal J for which $IJ = R$.

(2) Show that if R is a PID ("principal ideal domain") which is not a field, then R is a Dedekind domain. Conversely, show that if R is a Dedekind domain then R is a PID if and only if R is a UFD ("unique factorization domain").

(3) Let K be a number field, and for $\alpha \in K$ let $L_\alpha \colon K \to K$ be the linear transformation of $\mathbb{Q}$-vector spaces defined by $\beta \mapsto \alpha\beta$. Show that $|N_{K/\mathbb{Q}}(\alpha)| = |\det(L_\alpha)|$.

(4) Let K be a degree-n number field, let $\alpha \in \mathcal{O}_K$ satisfy $\mathbb{Q}(\alpha) = K$, and let $f(x)$ be the minimal polynomial for α over $\mathbb{Q}$. Write $\Delta(\alpha)$ for the discriminant $\Delta(1, \alpha, \dots, \alpha^{n-1})$ of the basis $1, \alpha, \alpha^2, \dots, \alpha^{n-1}$ of $K/\mathbb{Q}$. Let $\alpha_1, \dots, \alpha_n$ be the conjugates of α over $\mathbb{Q}$.

(a) Show that

$$\Delta(\alpha) = (-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} f'(\alpha_i) = (-1)^{\frac{n(n-1)}{2}} N_{K/\mathbb{Q}}(f'(\alpha)).$$

(b) Suppose that $f(x) = x^n + ax + b$. Show that

$$\Delta(\alpha) = (-1)^{\frac{n(n-1)}{2}} \Big((1-n)^{n-1} a^n + n^n b^{n-1} \Big).$$

Deduce that if $f(x) = x^2 + ax + b$ then $\Delta(\alpha) = a^2 - 4b$, and if $f(x) = x^3 + ax + b$ then $\Delta(\alpha) = -4a^3 - 27b^2$.