Math 676, Homework 6: due Oct 14

(1) Factor the ideals (2), (3), (7), (29), and (31) into prime ideals in $\mathbb{Z}[\sqrt[3]{2}]$.

(2) Let p be a prime, and let $c \in \mathbb{Z} \setminus \{0, 1, -1\}$ be a squarefree integer which is not divisible by p. Writing $\alpha := \sqrt[p]{c}$ and $K := \mathbb{Q}(\alpha)$, show that if $\mathcal{O}_K = \mathbb{Z}[\alpha]$ then $c^{p-1} \not\equiv 1 \pmod{p^2}$.
(Note: we showed the converse in class.)

(3) Determine the ideal class groups of $\mathbb{Z}[\sqrt{-14}]$, $\mathbb{Z}[\sqrt{-21}]$ and $\mathbb{Z}[\sqrt[3]{2}]$. You may use the following result of Minkowski's, which will be proved in class: if K is a degree-n number field and r_1 is the number of embeddings $K \hookrightarrow \mathbb{R}$, then every ideal class in $\mathcal{O}_K$ contains a nonzero ideal having norm at most
$$\frac{n!}{n^n} \cdot \left(\frac{4}{\pi}\right)^{\frac{n-r_1}{2}} \cdot \sqrt{|\Delta_K|}.$$