Math 676, Homework 7: due Oct 21

(1) Let $n \geq 3$ be an integer for which $p := 4n - 1$ is prime. Show that $\mathbb{Q}(\sqrt{-p})$ has class number 1 if and only if $x^2 + x + n$ is prime for each $x = 0, 1, 2, \ldots, n-2$. Find a monic quadratic $f(x) \in \mathbb{Z}[x]$ which takes prime values at 40 consecutive integers.

(2) For any nonnegative integers r_1 and r_2 which are not both zero, show that the set S is a convex subset of $\mathbb{R}^{r_1+2r_2}$, where S consists of all tuples $(a_1, \ldots, a_{r_1}, b_1, c_1, b_2, c_2, \ldots, b_{r_2}, c_{r_2})$ of real numbers satisfying

$$|a_1| + \cdots + |a_{r_1}| + 2\sqrt{b_1^2 + c_1^2} + \cdots + 2\sqrt{b_{r_2}^2 + c_{r_2}^2} \leq n.$$

(Here a set is called "convex" if whenever it contains two points, it also contains the line segment between them.)

(3) Show that if α is a nonzero algebraic integer such that every conjugate of α over $\mathbb{Q}$ is a complex number of absolute value 1, then α is a root of unity.

(Extra Credit) Let n be a positive integer. Suppose that some equilateral polygon in a plane has all angles being integral multiples of $\frac{\pi}{n}$, with the possible exception of two consecutive angles. Show that the remaining two angles are integral multiples of $\frac{\pi}{n}$ as well.