Midterm 1 review, Math 116

(1) FTC1: $\int_a^b F'(x)\,dx = F(b) - F(a)$

(2) FTC2: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$. More generally, $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$.

(3) Let R be the region in the x, y plane with $a \le x \le b$ and $f(x) \le y \le g(x)$, for some prescribed functions $f(x)$ and $g(x)$ such that $f(x) \le g(x)$ for $a \le x \le b$. Then the solid obtained by rotating R around the line $y = c$ has volume $\int_a^b \pi\big((f(x)-c)^2 - (g(x)-c)^2\big)\,dx$, and a small piece near $x = x_i$ has volume approximately $\pi\big((f(x_i)-c)^2 - (g(x_i)-c)^2\big)\Delta x$. The solid obtained by rotating R around the line $x = c$ has volume $\int_a^b \big(f(x) - g(x)\big)2\pi|x - c|\,dx$, and a small piece obtained by rotating the portion of R near $x = x_i$ has volume approximately $\big(f(x_i) - g(x_i)\big)2\pi|x_i - c|\,\Delta x$.

(4) If a solid S intersects the plane $x = x_0$ (in x, y, z space) in a cross-section with area $A(x_0)$, then the volume of the solid for $a \le x \le b$ is $\int_a^b A(x)\,dx$, and a small piece near $x = x_i$ has volume approximately $A(x_i)\Delta x$.

(5) The arc length of $y = f(x)$ from $x = a$ to $x = b$ is $\int_a^b \sqrt{1 + (f'(x))^2}\,dx$. *This must be positive – if it isn't then switch a and b.*

(6) Mass from density (§8.4, to be covered in class on Wednesday): find mass of one slice, then integrate over slices to get total mass. If density is constant then slice in any direction; if density depends on x then need dx in integral, so slice perpendicular to x-axis; if density depends on y then need dy in integral, so slice perpendicular to y-axis. Mass is density times volume.

(7) Be able to compute LEFT(n), RIGHT(n), MID(n), and TRAP(n) as estimates to an integral.

(8) Among LEFT(n), RIGHT(n), MID(n), TRAP(n), $\int_a^b f(x)\,dx$: if $f(x)$ is increasing on $[a, b]$ then LEFT(n) is the smallest of these five values, and RIGHT(n) is the biggest; if $f(x)$ is decreasing on $[a, b]$ then LEFT(n) is the biggest and RIGHT(n) is the smallest; if $f(x)$ is sometimes increasing and sometimes decreasing then in general we do not know how LEFT(n) and RIGHT(n) compare to the other three values.

(9) If $f(x)$ is concave up on $[a, b]$ then $\text{MID}(n) < \int_a^b f(x)\,dx < \text{TRAP}(n)$; if $f(x)$ is concave down on $[a, b]$ then $\text{MID}(n) > \int_a^b f(x)\,dx > \text{TRAP}(n)$.

(10) Keep in mind that $\int_a^b f(x)\,dx$ is the *signed* area between $y = f(x)$, $y = 0$, $x = a$, and $x = b$. This means you count the portion below the x-axis as negative.

(11) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ and $\int_a^b \big(f(x) + c\cdot g(x)\big)\,dx = \int_a^b f(x)\,dx + c\int_a^b g(x)\,dx$.

(12) If $f(x)$ is odd (i.e., $f(-x) = -f(x)$; think x^n with n odd) then $\int_{-a}^a f(x)\,dx = 0$.

(13) If $f(x)$ is even (i.e., $f(-x) = f(x)$; think x^n with n even) then $\int_{-a}^0 f(x)\,dx = \int_0^a f(x)\,dx$.

(14) The average value of $f(x)$ on the interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

(15) Antiderivatives: $\int f(x)\,dx$ (with no bounds a, b) is the collection of *all* antiderivatives of $f(x)$. It has the form $F(x) + C$ for any specific antiderivative $F(x)$ of $f(x)$, meaning that $F'(x) = f(x)$. *Don't forget the C.*

(16) Integration methods: if you see e^{x^3} or $\sin(x^3)$ or $\ln(x^3)$ then you'll usually substitute $u = x^3$. Likewise if you see $e^{\sin x}$ or $\cos(\sin x)$ or $\ln(\sin x)$ then you'll usually substitute $u = \sin x$. Basically you make a substitution that simplifies the expression you're trying to integrate. Don't forget to rewrite dx in terms of du, and to be careful that the bounds on the integral are x-values rather than u-values. Integration by parts: $\int u\,dv = uv - \int v\,du$. For this, choose dv to be a quantity that doesn't become much more complicated when we integrate it (best choices are e^x, $\sin x$, $\cos x$; next-best is x^n), and choose u to be a quantity that becomes simpler when we differentiate it (best choice is $\ln x$, next-best is x^n). Be sure you can integrate $\ln x$ (by parts, with $dv = 1$ and $u = \ln x$), and know that $\int x f'(x)\,dx = x f(x) - \int f(x)\,dx$ and $\int x f''(x)\,dx = x f'(x) - f(x) + C$.

(17) Partial fractions: to integrate $f(x) := \frac{ux+v}{(x-a)(x-b)}$ with $a \ne b$, find r, s so that $f(x) = \frac{r}{x-a} + \frac{s}{x-b}$, and then $\int f(x)\,dx = r\ln|x-a| + s\ln|x-b| + C$; to integrate $f(x) := \frac{ux^2+vx+w}{(x-a)^2(x-b)}$ with $a \ne b$, find r, s, t so that $f(x) = \frac{r}{(x-a)^2} + \frac{s}{x-a} + \frac{t}{x-b}$ **(and you should be able to find r, s, t in practice)**, and then $\int f(x)\,dx = \frac{-r}{x-a} + s\ln|x-a| + t\ln|x-b| + C$; to integrate $f(x) := \frac{ux^2+vw+w}{(x^2+a)(x-b)}$ with $a > 0$, find r, s, t so that $f(x) = \frac{rx+s}{x^2+a} + \frac{t}{x-b}$ *(but you'll never have to find r, s, t in practice)*, and then $\int f(x)\,dx = \frac{r}{2}\ln|x^2 + a| + \frac{s}{\sqrt{a}}\arctan(\frac{x}{\sqrt{a}}) + t\ln|x-b| + C$.

(18) The maximum and minimum of a function $f(x)$ on an interval $[a, b]$ always occur at either a or b or at some c satisfying $f'(c) = 0$. If $f'(c) = 0$ then c can be a local max, a local min, or neither of these (e.g. $c = 0$ for $f(x) = x^3$).

(19) If you're drawing the graph of a function $f(x)$, keep in mind where $f(x)$ is positive, where $f'(x)$ is positive (i.e., $f(x)$ is increasing), and where $f''(x)$ is positive (i.e., $f(x)$ is concave up). Make sure that your graph is continuous and has no sharp corners, and that if you know the value $f(c)$ for some c then your graph passes through the point $(c, f(c))$.