On my honor, as a student,
I have neither given nor received
unauthorized aid on this academic work. **Initials:** MZ

Do not write in this area

Math 116 — Second Midterm — November 13, 2017

Your Initials Only: MZ Your U-M ID # (not uniqname): ______

Instructor Name: Zieve Section #: ______

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 10 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is <u>not</u> permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is <u>not</u>.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	12	
2	9	
3	12	
4	13	
5	5	

Problem	Points	Score
6	7	
7	12	
8	9	
9	12	
10	9	
Total	100	

1. [12 points] Consider the infinite sequences c_n, d_n, j_n and ℓ_n, defined for $n \geq 1$ as follows:

$$c_n = \sum_{k=1}^{n} \frac{(-1)^k}{k!}$$

$$d_n = \arctan(1.1^n)$$

$$j_n = \int_0^{n^3} e^{2x}\,dx$$

$\ell_n = \sin(x^n)$ for some fixed value of x satisfying $0 < x < 1$.

a. [8 points] Decide whether each of these sequences is bounded, unbounded, always increasing, and/or always decreasing. Record your conclusions by clearly circling the correct descriptions below. Contradictory conclusions will be marked incorrect.

i. The sequence c_n is

Since $\sum_{k=1}^{\infty} \frac{(-1)^k}{k!}$ converges by AST (but alternating, so c_n isn't increasing or decreasing)

bounded unbounded increasing decreasing

ii. The sequence d_n is

1.1^n is increasing with limit ∞, so d_n is increasing with limit $\frac{\pi}{2}$.

bounded ~~unbounded~~ increasing decreasing

iii. The sequence j_n is

$j_n - j_{n-1} = \int_{(n-1)^3}^{n^3} e^{2x}dx$ is positive since $n^3 > (n-1)^3$ and $e^{2x} > 0$, so j_n is increasing.

bounded unbounded increasing decreasing

$\lim_{n\to\infty} j_n = \lim_{b\to\infty} \int_0^b e^{2x}dx = \lim_{b\to\infty} \frac{e^{2x}}{2}\Big]_0^b = \lim_{b\to\infty} \frac{(e^{2b}-1)}{2} = \infty$.

iv. The sequence ℓ_n is

bounded unbounded increasing decreasing

x^n is decreasing with limit 0 and biggest value less than 1, so ℓ_n is decreasing with limit $\sin(0)=0$.

b. [4 points] For parts i and ii below, decide whether the sequence converges or diverges.

- If the sequence converges, circle "converges", find the value to which it converges, and write this value on the answer blank provided.
- If the sequence diverges, circle "diverges".

i. The sequence d_n **Converges to** $\pi/2$ **Diverges**

ii. The sequence j_n **Converges to** __________ **Diverges**

2. [9 points] Consider the graph of $y = e^{-14x}$ for $x \geq 0$.

a. [3 points] Let $\mathcal{R}$ be the region in the first quadrant between the graph of $y = e^{-14x}$ and the x-axis. Which of the following improper integrals best expresses the volume of the solid that is obtained by rotating $\mathcal{R}$ around the x-axis?

Circle one:

$\int_0^{\infty} \pi e^{-14x}\,dx$ $\qquad$ $\int_0^{\infty} x e^{-14x}\,dx$ $\qquad$ $\int_0^{\infty} x e^{-28x}\,dx$ $\qquad$ $\int_0^{\infty} \pi e^{-28x}\,dx$

$\frac{\pi}{7}\int_0^1 \ln(y)\,dy$ $\qquad$ $\frac{\pi}{14}\int_0^1 y\ln(y)\,dy$ $\qquad$ $\frac{\pi}{14}\int_0^1 y\ln(y)\,dy$

b. [6 points] Determine whether the improper integral you circled in part **a** converges or diverges.

- If the integral converges, circle "converges", find its exact value (i.e. no decimal approximations), and write the exact value on the answer blank provided.
- If the integral diverges, circle "diverges" and justify your answer.

In either case, **you must show all your work carefully using correct notation.** Any direct evaluation of integrals must be done **without using a calculator.**

Converges to $\pi/28$ $\qquad\qquad$ **Diverges**

$$\int_0^{\infty} \pi e^{-28x}\,dx = \lim_{b\to\infty} \int_0^b \pi e^{-28x}\,dx = \lim_{b\to\infty} \frac{\pi e^{-28x}}{-28}\Big|_0^b$$

$$= \lim_{b\to\infty}\left(\frac{\pi e^{-28b}}{-28} - \frac{\pi}{-28}\right) = \frac{\pi}{28}$$

3. [12 points] Define a sequence a_n for $n \geq 1$ by $a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1/3 & \text{if } n \text{ is even.} \end{cases}$

Note that this means the first three terms of this sequence are $a_1 = 0$, $a_2 = 1/3$, $a_3 = 0$.

a. [2 points] Does the sequence $\{a_n\}$ converge or diverge?

Circle one: **converges** **diverges**

Briefly explain your answer.

$\lim_{n\to\infty} a_{2n} = \frac{1}{3} \neq 0 = \lim_{n\to\infty} a_{2n+1}$, so there is no single value which is arbitrarily close to a_n for all big n.

b. [4 points] Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(3 \cdot \overset{=1/3}{a_{2n}})}{n3^n} x^n$.

i. Write out the partial sum with three terms for this power series.

Answer: $\frac{1}{3}x + \frac{1}{2\cdot 3^2}x^2 + \frac{1}{3\cdot 3^3}x^3$

ii. Give the exact value of this partial sum when $x = -\frac{1}{2}$.

Answer: $\frac{1}{3}\cdot\frac{-1}{2} + \frac{1}{2\cdot 3^2}\cdot\frac{1}{4} + \frac{1}{3\cdot 3^3}\cdot\frac{-1}{8}$

c. [6 points] Determine the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \overset{=a_n}{\frac{x^n}{n \cdot 3^n}}$.

Show every step of any calculations and fully justify your answer with careful reasoning. Write your final answer on the answer blank provided.

Centered at 0. For $x \neq 0$, $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{(n+1)\cdot 3^{n+1}}}{\frac{x^n}{n\cdot 3^n}}\right| = \frac{|x|}{3}\cdot\frac{n}{n+1}$,

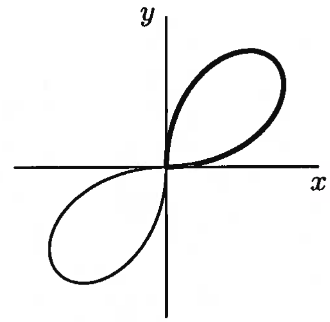
So $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{3}$, which is less than 1 precisely when $|x| < 3$.

Hence (by Ratio Test) the radius of convergence is 3. When $x = 3$ the series is $\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges (p-test, $p=1$). When $x=-3$ the series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ which converges by AST, since $\frac{1}{n+1} < \frac{1}{n}$ for $n \geq 1$ and $\lim_{n\to\infty}\frac{1}{n} = 0$.

Answer: $[-3, 3)$

4. [13 points] The orbit of a single electron around the nucleus of an atom is determined by the energy level of that electron and by the other electrons orbiting the nucleus. We can model one electron's orbital in two-dimensions as follows. Suppose that the nucleus of an atom is centered at the origin. Then the (so-called "$2p_1$") orbital has the shape shown below.

This shape is made up of two regions that we call "lobes". The outer edge of the lobes are described by the polar equation $r = k\sin(2\theta)$ for some positive constant k. Note that only the relevant portion of the polar curve $r = k\sin(2\theta)$ is shown.
The "top lobe" is the portion in the first quadrant (shown in bold).



a. [2 points] For what values of θ with $0 \le \theta \le 2\pi$ does the polar curve $r = k\sin(2\theta)$ pass through the origin?

Need $k\sin(2\theta)=0$, so $\sin(2\theta)=0$, so $\theta = \boxed{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \text{ or } 2\pi.}$

b. [3 points] For what values of θ does the polar curve $r = k\sin(2\theta)$ trace out the "top lobe"? Give your answer as an interval of θ values.

$0 \le \theta \le \frac{\pi}{2}$

c. [4 points] Write, but do **not** evaluate, an integral that gives the area of the top lobe.

$$\int_0^{\frac{\pi}{2}} \frac{1}{2}\left(k\sin(2\theta)\right)^2 d\theta$$

d. [4 points] Imagine that an electron lies within the top lobe of this orbital, but is as far away from the origin as possible. What are the polar coordinates of this point of greatest distance from the origin? Your answer may involve the constant k.

The biggest value of r occurs when $\sin(2\theta)=1$, which for $0 \le \theta \le \frac{\pi}{2}$ means $\theta = \frac{\pi}{4}$, so $\boxed{r = k, \ \theta = \frac{\pi}{4}}$

5. [5 points] Consider the improper integral $\displaystyle\int_1^\infty \frac{2}{3x+5e^x}\,dx$.

Note that for $x > 0$, we have $\displaystyle\frac{2}{3x+5e^x} < \frac{2}{3x}$ and $\displaystyle\frac{2}{3x+5e^x} < 0.4e^{-x}$.

Use this information together with the (Direct) Comparison Test for Integrals to determine whether $\displaystyle\int_1^\infty \frac{2}{3x+5e^x}\,dx$ converges or diverges.

Write the comparison function you use on the blank below and circle your conclusion for the improper integral. Then briefly explain your reasoning.

Answer: Using (direct) comparison of $\displaystyle\frac{2}{3x+5e^x}$ with the function $0.4e^{-x}$,

the improper integral $\displaystyle\int_1^\infty \frac{2}{3x+5e^x}\,dx$ **Converges** **Diverges**

Briefly explain your reasoning.

$\displaystyle\int_1^\infty 0.4e^{-x}\,dx = \lim_{b\to\infty}\int_1^b 0.4e^{-x}\,dx = \lim_{b\to\infty} -0.4e^{-x}\Big]_1^b = \lim_{b\to\infty}\left(-0.4e^{-b}+0.4e^{-1}\right) = \frac{0.4}{e}$ converges,

So since $0 < \displaystyle\frac{2}{3x+5e^x} < 0.4e^{-x}$ for $x \geq 1$ it follows by Comp. Test

that $\displaystyle\int_1^\infty \frac{2}{3x+5e^x}\,dx$ converges.

6. [7 points] Consider the series $\displaystyle\sum_{n=2}^\infty \frac{n^2-n+2}{4n^4-3n^2}$. $=: a_n$

Use the **Limit Comparison Test** to determine whether this series converges or diverges. Circle your answer (either "converges" or "diverges") clearly.

The series $\displaystyle\sum_{n=2}^\infty \frac{n^2-n+2}{4n^4-3n^2}$ **Converges** **Diverges**

Give full evidence to support you answer below. Be sure to clearly state your choice of comparison series, show each step of any computation, and carefully justify your conclusions.

Put $b_n := \frac{1}{n^2}$, so $a_n, b_n > 0$ for $n \geq 2$ and

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^4-n^3+2n^2}{4n^4-3n^2} = \lim_{n\to\infty}\frac{1-\frac{1}{n}+\frac{2}{n^2}}{4-\frac{3}{n^2}} = \frac{1}{4}$$ is finite and

nonzero, whence by LCT $\displaystyle\sum_{n=2}^\infty a_n$ converges because

$\displaystyle\sum_{n=2}^\infty b_n$ converges (which is true by p-test with $p=2$).

7. [12 points] A bouncy ball is launched up 20 feet from the floor and then begins bouncing. Each time the ball bounces up from floor, it bounces up again to a height that is 60% the height of the previous bounce. (For example, when it bounces up from the floor after falling 20 ft, the ball will bounce up to a height of $0.6(20) = 12$ feet.)
Consider the following sequences, defined for $n \geq 1$:

- Let h_n be the height, in feet, to which the ball rises when the ball leaves the ground for the nth time. So $h_1 = 20$ and $h_2 = 12$
- Let f_n be the total distance, in feet, that the ball has traveled (both up and down) when it bounces on the ground for the nth time. For example, $f_1 = 40$ and $f_2 = 40 + 24 = 64$.

a. [2 points] Find the values of h_3 and f_3.

Answer: $h_3 =$ $12\cdot(0.6)$ and $f_3 =$ $64 + 24\cdot(0.6)$

b. [6 points] Find a closed form expression for h_n and f_n.
("Closed form" here means that your answers should not include sigma notation or ellipses $(\cdots)$. Your answers should also **not** involve recursive formulas.)

$$h_n = 20\cdot(0.6)^{n-1}$$

$$f_n = \sum_{k=1}^{n} 2h_k = \sum_{k=1}^{n} 40\cdot(0.6)^{k-1} = \sum_{k=0}^{n-1} 40\cdot(0.6)^{k}$$

$$= 40\cdot\frac{1-(0.6)^n}{1-0.6}$$

Answer: $h_n =$ $20\cdot(0.6)^{n-1}$ and $f_n =$ $40\cdot\frac{1-(0.6)^n}{1-0.6}$

c. [4 points] Decide whether the given sequence or series converges or diverges.
If it diverges, circle "diverges". If it converges, circle "converges" and write the value to which it converges in the blank.

i. The sequence f_n

Converges to $\frac{40}{1-0.6}$ **Diverges**

ii. The series $\sum_{n=1}^{\infty} h_n$

Converges to $\frac{20}{1-0.6}$ **Diverges**

8. [9 points] For each of parts **a** through **c** below, circle all of the statements that <u>must be true</u>. Circle "NONE OF THESE" if none of the statements must be true.
You must circle at least one choice to receive any credit.
No credit will be awarded for unclear markings. No justification is necessary.

a. [3 points] Suppose $f(x)$ is a continuous and decreasing function on the interval $[0,2]$ with $f(0) = 1$ and $f(2) = 0$.

Let a be a constant with $0 < a < 1$. Consider the integral $\displaystyle\int_a^2 \frac{1}{f(x)}\,dx$.

i. This integral is not improper. No - when $x \to 2^-$, $\frac{1}{f(x)} \to \infty$

ii. This integral converges by direct comparison with the constant function 1. This means $\frac{1}{f(x)} < 1$, so $1 < f(x)$, but $f(x)$ is decreasing from 1 to 0.

iii. This integral converges by direct comparison with the function $f(x)$. This means $\frac{1}{f(x)} < f(x)$, so $1 < f(x)^2$, False as above.

iv. This integral converges for some values of a between 0 and 1 but diverges for other values of a between 0 and 1. Need not be true - for instance, if $f(x) = 1 - \frac{x}{2}$ then, for $0<a<1$, $\int_a^2 \frac{1}{f(x)}dx = \lim_{b\to 2^-} \int_a^b \frac{1}{1-\frac{x}{2}}dx$ $\lim_{b\to2^-} \left.\frac{\ln(1-\frac{x}{2})}{-\frac{1}{2}}\right]_a^b = \infty$ diverges

v. NONE OF THESE

b. [3 points] Suppose $g(x)$ is a positive and decreasing function that is defined and continuous on the open interval $(5, \infty)$ such that

$$\int_{10}^{\infty} g(x)\,dx \ \underline{\text{converges}} \qquad \text{and} \qquad \int_5^8 g(x)\,dx \ \underline{\text{diverges}}.$$

i. The series $\displaystyle\sum_{n=20}^{\infty} g(n)$ converges. By integral test, since $g(x)$ positive + decreasing for $x>20$ and $\int_{20}^{\infty} g(x)\,dx$ converges.

ii. The series $\displaystyle\sum_{n=12}^{\infty} \frac{1}{g(n)}$ diverges. By nth term test, since the convergence in (i) implies $\lim_{n\to\infty} g(n) = 0$ so $\lim_{n\to\infty} \frac{1}{g(n)} \neq 0$.

iii. The sequence $c_n = \displaystyle\int_{15}^{n} g(x)\,dx$, $n \geq 15$, converges. Because $\int_{10}^{\infty} g(x)dx$ converges so $\lim_{n\to\infty} c_n = \int_{15}^{\infty} g(x)dx$ also converges.

iv. The integral $\displaystyle\int_5^7 g(x)\,dx$ diverges. Because $\int_7^8 g(x)\,dx$ is a definite integral, and hence is finite, and $\int_5^8 g(x)dx$ diverges, so $\int_5^7 g(x)dx$ diverges.

v. NONE OF THESE

c. [3 points] Consider the sequence $a_n = \dfrac{1}{\ln(n)}$, $n \geq 2$. note: $n \geqslant 2$ here. So change "$n=1$" to "$n=2$" in parts ii, iii, iv in order to get intelligible questions.

i. $\displaystyle\lim_{n\to\infty} a_n = 0$. Since $\lim_{n\to\infty} \ln(n) = \infty$.

ii. The series $\displaystyle\sum_{n=\cancel{1}\,2}^{\infty} a_n$ converges.

iii. The series $\displaystyle\sum_{n=\cancel{1}\,2}^{\infty} a_n$ diverges. By Comp. Test, since $a_n > \frac{1}{n} > 0$ for $n \geqslant 2$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by p-test ($p=1$).

iv. The series $\displaystyle\sum_{n=\cancel{1}\,2}^{\infty} (-1)^n a_n$ converges. By AST, since $a_n > a_{n+1} > 0$ for $n \geqslant 2$ and $\lim_{n\to\infty} a_n = 0$.

v. NONE OF THESE

9. [12 points] For each of parts **a** through **c** below, determine the radius of convergence of the power series. Show your work carefully.

a. [3 points] $\displaystyle\sum_{n=1}^{\infty} \frac{e}{n!}(x-1)^n \;=: a_n$

For $x \neq 1$, $\left|\frac{a_{n+1}}{a_n}\right| = \left|\dfrac{\frac{e}{(n+1)!}\cdot(x-1)^{n+1}}{\frac{e}{n!}\cdot(x-1)^n}\right| = \dfrac{|x-1|}{n+1}$, so $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$.

$\therefore$ By the Ratio Test the series converges for all values of x.

Answer: radius of convergence = ∞

b. [3 points] $5(x+\pi) + 5\cdot 4(x+\pi)^2 + 5\cdot 9(x+\pi)^3 + 5\cdot 16(x+\pi)^4 + \cdots \;= \displaystyle\sum_{n=1}^{\infty} \underbrace{5\cdot n^2\cdot(x+\pi)^n}_{a_n}$

For $x \neq -\pi$, $\left|\frac{a_{n+1}}{a_n}\right| = \left|\dfrac{5\cdot(n+1)^2\cdot(x+\pi)^{n+1}}{5\cdot n^2\cdot(x+\pi)^n}\right| = \left(\frac{n+1}{n}\right)^2\cdot|x+\pi|$,

so $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x+\pi|$, whence by the Ratio Test the series converges if $|x+\pi|<1$ and diverges if $|x+\pi|>1$.

Answer: radius of convergence = 1

c. [3 points] $\displaystyle\sum_{n=0}^{\infty} \frac{\pi}{8^n}(x+2)^{3n} \;=: a_n$

For $x \neq -2$, $\left|\frac{a_{n+1}}{a_n}\right| = \left|\dfrac{\frac{\pi}{8^{n+1}}\cdot(x+2)^{3(n+1)}}{\frac{\pi}{8^n}\cdot(x+2)^{3n}}\right| = \dfrac{|x+2|^3}{8}$,

So $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \dfrac{|x+2|^3}{8}$, whence by the Ratio Test the series converges if $\left|\frac{(x+2)^3}{8}\right|<1$, i.e., $|x+2|<2$, and diverges if $|x+2|>2$.

Answer: radius of convergence = 2

d. [3 points] Consider the power series $\sum_{j=0}^{\infty} C_j(x-5)^j$, where each C_j is a constant. Suppose this power series

- converges when $x = 2$ and
- diverges when $x = 12$.

Radius of convergence is at least $|5-2|=3$ and at most $|5-12|=7$.

Based on this information, which of the following values **could** be equal to the radius of convergence of the power series? Circle all possibilities from the list below.

0	1	2	3	4
5	6	7	8	9
10	11	12	NONE OF THESE	

10. [9 points] The sequence $\{\gamma_n\}$ is defined according to the formula

$$\gamma_n = -\ln(n) + \sum_{k=1}^{n} \frac{1}{k}.$$

(You may recall this sequence from team homework 5.) This sequence converges to a positive number γ (which happens to be $\gamma \approx 0.5772156649$).

a. [2 points] Does the sequence $\{\gamma_n^2\}$ converge or diverge? If this sequence converges, compute the value to which this sequence converges, either in terms of the constant γ or with five decimal places of accuracy.

$$\lim_{n\to\infty} \gamma_n^2 = \left(\lim_{n\to\infty} \gamma_n\right)^2 = \boxed{\gamma^2}$$

Converges

b. [3 points] Does the series $\sum_{n=1}^{\infty} \gamma_n$ converge or diverge? Briefly explain your answer, and if this series converges, compute the value to which the series converges either in terms of the constant γ or with five decimal places of accuracy.

Diverges by n^{th} term test, since $\lim_{n\to\infty} \gamma_n = \gamma \neq 0$.

c. [4 points] Let $h_n = \sum_{k=1}^{n} \frac{1}{k}$. Find the value of $\lim_{n\to\infty} \frac{e^{h_n}}{n}$.

You may give your answer either in terms of the constant γ or with five decimal places of accuracy.

Hint: First consider $\lim_{n\to\infty} \ln\left(\frac{e^{h_n}}{n}\right)$.

$$\lim_{n\to\infty} \ln\left(\frac{e^{h_n}}{n}\right) = \lim_{n\to\infty} \left(\ln(e^{h_n}) - \ln(n)\right) = \lim_{n\to\infty} \left(h_n - \ln(n)\right)$$

$$= \lim_{n\to\infty} \gamma_n = \gamma$$

So $\lim_{n\to\infty} \frac{e^{h_n}}{n} = e^{\ln\left(\lim_{n\to\infty} \frac{e^{h_n}}{n}\right)} = e^{\lim_{n\to\infty} \ln\left(\frac{e^{h_n}}{n}\right)}$

$= \boxed{e^{\gamma}}$