Math 116 — Practice for Exam 1

Generated January 22, 2023

Name: SOLUTIONS

Instructor: ______________________ Section Number: ________

1. This exam has 4 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2016	1	1	catapult	16	
Winter 2014	1	1		7	
Winter 2014	1	2		5	
Winter 2015	1	4		14	
Total				42	

Recommended time (based on points): 38 minutes

1. [16 points] At a time t seconds after a catapult throws a rock, the rock has horizontal velocity $v(t)$ m/s. Assume $v(t)$ is monotonic between the values given in the table and does not change concavity.

t	0	1	2	3	4	5	6	7	8
$v(t)$	47	34	24	16	10	6	3	1	0

a. [4 points] Estimate the average horizontal velocity of the rock between $t = 2$ and $t = 5$ using the trapezoid rule with 3 subdivisions. Write all the terms in your sum. Include **units**.

Solution:

$$\frac{\int_2^5 v(t)\,dt}{5-2} = \frac{Left(3) + Right(3)}{2 \cdot 3} = \frac{(v(2)+v(3)+v(4)) + (v(3)+v(4)+v(5))}{6} =$$

$$= \frac{24+16+10+16+10+6}{6} = \frac{82}{6} = \frac{41}{3}$$

The average horizontal velocity of the rock is 41/3 m/s.

b. [4 points] Estimate the total horizontal distance the rock traveled using a left Riemann sum with 8 subdivisions. Write all the terms in your sum. Include **units**.

Solution:

$$\int_0^8 v(t)\,dt = Left(8) = v(0)+v(1)+v(2)+v(3)+v(4)+v(5)+v(6)+v(7) =$$

$$= 47+34+24+16+10+6+3+1 = 141$$

The total horizontal distance the rock traveled is approximately 141 meters.

c. [4 points] Estimate the total horizontal distance the rock traveled using the midpoint rule with 4 subdivisions. Write all the terms in your sum. Include **units**.

Solution:

$$\int_0^8 v(t)\,dt = Mid(4) = 2(v(1)+v(3)+v(5)+v(7)) =$$

$$= 2(34+16+6+1) = 114$$

The total horizontal distance the rock traveled is approximately 114 meters.

d. [4 points] A second rock thrown by the catapult traveled horizontally 125 meters. Determine whether the first rock or the second rock traveled farther, or if there is not enough information to decide. Circle your answer. Justify your answer.

Solution:

the first rock $\qquad$ $\boxed{\text{the second rock}}$ $\qquad$ not enough information

The function $v(t)$ is concave up since for example

$$-10 = \frac{v(2)-v(1)}{2-1} > \frac{v(1)-v(0)}{1-0} = -13$$

The trapezoid rule gives $Trap(4) = 121$ (or $Trap(8) = 117.5$). Since $v(t)$ is concave up, this is an overestimate which means that the first rock traveled at most 121 meters, that is less than the second.

1. [7 points] The table below gives values of a function, $f(x)$, at several points.

x	4	5	6	7	8
$f(x)$	3	5	4	1	2

a. [3 points] Estimate the integral $\int_4^8 f(x)dx$ using Mid(2). Be sure to write out all the terms of your sum.

Solution:
Mid(2) $= 2(f(5) + f(7)) = 2(5 + 1) = 12$.

b. [4 points] Simplify the integral $\displaystyle\int_{\ln(4)}^{\ln(7)} e^x f(e^x)dx$ and estimate the resulting integral using Trap(3). Be sure to show how you simplified the integral and to write out all the terms of your sum.

Solution:
Let $u = e^x$ then $du = e^x dx$. Changing the bounds of integration upper bound$= e^{\ln(7)} = 7$, lower bound$= e^{\ln(4)} = 4$. Thus $\displaystyle\int_{\ln(4)}^{\ln(7)} e^x f(e^x)dx = \int_4^7 f(u)du$.
Trap(3) $= \frac{1}{2}$(Left(3)+Right(3)) $= \frac{1}{2}f(4) + f(5) + f(6) + \frac{1}{2}f(7) = 11$.

2. [5 points] Suppose that $g(x) = w(x)v(x)$ where the functions $w(x)$ and $v(x)$ are both positive, decreasing and concave down on the interval $[0, 1]$.

a. [2 points] Write the derivatives $g'(x)$ and $g''(x)$ in terms of $w(x)$, $v(x)$, and their derivatives.

Solution:
$g'(x) = w'(x)v(x) + w(x)v'(x)$
$g''(x) = w''(x)v(x) + 2w'(x)v'(x) + w(x)v''(x)$

b. [3 points] Circle the method(s) that will ALWAYS UNDERESTIMATE the integral $\int_0^1 g(x)dx$.

Left $\qquad$ $\boxed{\text{Right}}$ $\qquad$ Mid $\qquad$ Trap

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Left $\qquad$ $\boxed{\text{Right}}$ $\qquad$ Mid $\qquad$ Trap

4. [14 points] The function

$$f(x) = \sin(\sqrt{x})$$

does not have an antiderivative that can be written in terms of elementary functions. However, we can use the second fundamental theorem of calculus to construct an antiderivative for f. We define an antiderivative F of f by

$$F(x) = \int_0^x \sin(\sqrt{t})dt.$$

a. [2 points] The concavity of F does not change on the interval $\left(0, \frac{\pi^2}{4}\right)$. Determine the concavity of F on $\left(0, \frac{\pi^2}{4}\right)$ and circle one of the options below. No justification is needed.

Solution:

$\boxed{\textbf{Concave Up}}$ **Concave Down** **Neither**

b. [2 points] Using the blanks provided, order from least to greatest

$$F\left(\frac{\pi^2}{4}\right), \quad \mathrm{LEFT}\,(100), \quad \mathrm{RIGHT}\,(100), \quad \mathrm{MID}\,(100), \quad \mathrm{TRAP}\,(100),$$

where all the approximations are of the definite integral given by $F\left(\frac{\pi^2}{4}\right)$. No justification is needed.

Solution: LEFT (100) $\leq$ TRAP (100) $\leq$ $F\left(\frac{\pi^2}{4}\right)$ $\leq$ MID (100) $\leq$ RIGHT (100)

c. [4 points] Write out, but do not compute, MID (3) to approximate $F\left(\frac{\pi^2}{4}\right)$.

Solution: $\mathrm{MID}\,(3) = \left[\sin\left(\sqrt{\frac{\pi^2}{24}}\right) + \sin\left(\sqrt{\frac{3\pi^2}{24}}\right) + \sin\left(\sqrt{\frac{5\pi^2}{24}}\right)\right]\left(\frac{\pi^2}{12}\right)$

d. [4 points] Write out, but do not compute, TRAP (3) to approximate $F\left(\frac{\pi^2}{4}\right)$.

Solution: We have $\mathrm{LEFT}\,(3) = \left(\sin(0) + \sin\left(\sqrt{\frac{\pi^2}{12}}\right) + \sin\left(\sqrt{\frac{2\pi^2}{12}}\right)\right)\left(\frac{\pi^2}{12}\right)$,
and $\mathrm{RIGHT}\,(3) = \left(\sin\left(\sqrt{\frac{\pi^2}{12}}\right) + \sin\left(\sqrt{\frac{2\pi^2}{12}}\right) + \sin\left(\sqrt{\frac{3\pi^2}{12}}\right)\right)\left(\frac{\pi^2}{12}\right)$.
Then $\mathrm{TRAP}\,(3) = \frac{\mathrm{LEFT}(3)+\mathrm{RIGHT}(3)}{2}$.

e. [2 points] If you want to approximate $F\left(\frac{\pi^2}{4}\right)$ using right and left sums, what is the smallest number of subdivisions, n, you would have to use to guarantee that the difference between LEFT(n) and RIGHT(n) is less than or equal to 0.005?

Solution:

$$\text{RIGHT}(n) - \text{LEFT}(n) < 0.005 \iff \left(\sin\left(\sqrt{\frac{\pi^2}{4}}\right) - \sin(0)\right)\frac{\frac{\pi^2}{4} - 0}{n} < 0.005$$

$$\iff n > \frac{\pi^2}{0.02} \approx 493.$$

Since we need an integer number of subdivisions, we take $n = 494$.