Math 116 — Practice for Exam 3

Generated April 6, 2023

Name: SOLUTIONS

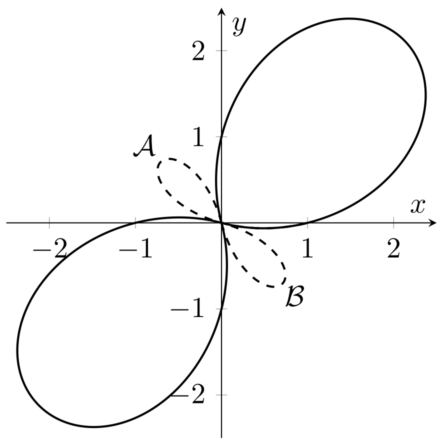
Instructor: ______________________ Section Number: __________

1. This exam has 4 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2022	3	5		16	
Fall 2021	3	7	fan	17	
Winter 2021	3	7	boats	15	
Fall 2019	3	2	rodent bite	9	
Total				57	

Recommended time (based on points): 68 minutes

5. [16 points] The following problems relate to the polar graph shown below, defined by the polar curve $r(\theta) = 2\sin(2\theta) + 1$, on the domain $[0, 2\pi]$. Both the dashed and solid curves are part of the graph of $r(\theta)$.



a. [4 points] Find all θ values in the interval $[0, 2\pi]$ such that $r(\theta) = 0$.

Solution: We set $r(\theta) = 2\sin(2\theta) + 1 = 0$ and solve. This becomes:

$$\sin(2\theta) = \frac{-1}{2}$$

This means $2\theta = \frac{-\pi}{6} + 2k\pi$ or $\frac{7\pi}{6} + 2k\pi$ Now if we divide by 2, we get $\theta = \frac{-\pi}{12} + k\pi$ or $\frac{7\pi}{12} + k\pi$. Now, since we need to be in the interval$[0, 2]$, we take $k = 1, 2$ for the first possible term to get:

$$\theta = \frac{-\pi}{12} + \pi = \frac{11\pi}{12} \qquad \theta = \frac{-\pi}{12} + 2\pi = \frac{23\pi}{12}.$$

We take $k = 0, 1$ for the second possible term to get:

$$\theta = \frac{7\pi}{12} + 0\pi = \frac{7\pi}{12} \qquad \theta = \frac{7\pi}{12} + \pi = \frac{19\pi}{12}$$

as answers

Answers: $\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

b. [4 points] Determine the θ intervals corresponding to the dashed portions $\mathcal{A}$ and $\mathcal{B}$ of the curve above.

Solution: We need to examine when our radius is changing sign. First, we see that at $\theta = 0$, the radius is positive. It must stay positive, sweeping out the large undotted portion in the first quadrant, until $\theta = \frac{7\pi}{12}$, which is when the radius is first zero, based on our work in a). At this point, the radius becomes negative. The next zero occurs at $\frac{11\pi}{12}$, which means that for the interval $[\frac{7\pi}{12}, \frac{11\pi}{12}]$, the radius is negative, which should sweep out the dotted curve $\mathcal{B}$. Continuing in this way, we see that the next interval $[\frac{11\pi}{12}, \frac{19\pi}{12}]$ will correspond to the next solid region, and then the interval $[\frac{19\pi}{12}, \frac{23\pi}{12}]$, we trace out the curve $\mathcal{A}$.

Interval for $\mathcal{A}$: $[\frac{19\pi}{12}, \frac{23\pi}{12}]$ Interval for $\mathcal{B}$: $[\frac{7\pi}{12}, \frac{11\pi}{12}]$

c. [4 points] Write an expression involving one or more integrals for the area of the region enclosed by the **solid** curves only (do not include the region enclosed by the dashed curves).

Solution: The solid portions are where the dashed portions are not. Therefore they are defined by the angles which are not our answers in b). Since we still need to be in $[0, 2\pi]$, This means our answers are:

$$\frac{1}{2}\int_0^{\frac{7\pi}{12}} (2\sin(2\theta)+1)^2\,d\theta + \frac{1}{2}\int_{\frac{11\pi}{12}}^{\frac{19\pi}{12}} (2\sin(2\theta)+1)^2\,d\theta + \frac{1}{2}\int_{\frac{23\pi}{12}}^{2\pi} (2\sin(2\theta)+1)^2\,d\theta$$

d. [4 points] Write an expression involving one or more integrals for the total arc length of the **dashed** curves in the graph above.

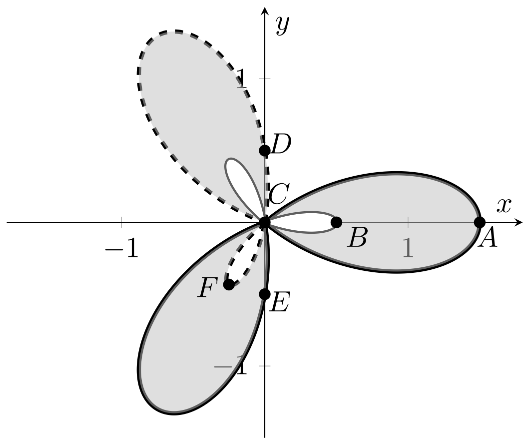
Solution: We can do this two ways. The first, is to have one integral for each curve. Using the polar arclength formula we get:

$$\int_{\frac{7\pi}{12}}^{\frac{11\pi}{12}} \sqrt{(2\sin(\theta)+1)^2 + (4\cos(2\theta))^2}d\theta + \int_{\frac{19\pi}{12}}^{\frac{23\pi}{12}} \sqrt{(2\sin(\theta)+1)^2 + (4\cos(2\theta))^2}d\theta$$

The second is to double one integral. Using the polar arclength formula we get:

$$2\int_{\frac{7\pi}{12}}^{\frac{11\pi}{12}} \sqrt{(2\sin(\theta)+1)^2 + (4\cos(2\theta))^2}d\theta$$

7. [17 points] John is holding a Fan Fair to celebrate the success of his burgeoning fan business. At the fair, John is debuting his new fan, which has blades given by the shaded region of the graph of the polar equation $r = \cos(3\theta) + \frac{1}{2}$ shown below. Note that the graph of $r = \cos(3\theta) + \frac{1}{2}$ is comprised of both the inner and outer loops of the fan blades. One of the activities at the Fan Fair is to guess the perimeter and area of the blades, which can actually be computed explicitly.



a. [4 points] For the values of θ listed below, write on the line the letter of the point corresponding to it.

$\theta = 0$: A $\theta = \frac{\pi}{3}$: F

$\theta = \frac{\pi}{2}$: D $\theta = \pi$: B

b. [5 points] Find the 3 values of θ which correspond to the point C (the origin) for $0 \le \theta \le \pi$. Then, determine the interval within $[0, 2\pi]$ for which θ traces out the **dashed** loops in the graph above. (*Hint:* $\cos\frac{2\pi}{3} = \cos\frac{4\pi}{3} = -\frac{1}{2}$)

Solution: Note that the solutions to $\cos(3\theta) = -\frac{1}{2}$ are, for k an integer

$$3\theta = \frac{2\pi}{3} + 2\pi k$$
$$3\theta = \frac{4\pi}{3} + 2\pi k.$$

So, the first three values are

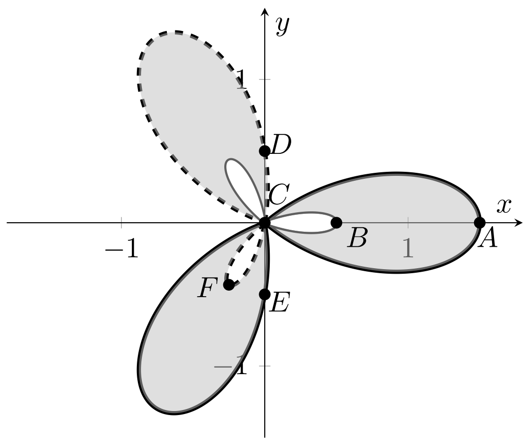
$$3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}.$$

Using (a), we know that the dashed loops lie between the first value and the third value when θ corresponds to the point C.

$\theta =$ $\frac{2\pi}{9}$, $\frac{4\pi}{9}$, $\frac{8\pi}{9}$

Interval giving θ-values that trace out the dashed loops: $[\frac{2\pi}{9}, \frac{8\pi}{9}]$

7. (continued) Here is a reproduction of the graph from the previous page of the polar equation $r = \cos(3\theta) + \frac{1}{2}$:



c. [4 points] Write, but do not evaluate, an expression involving one or more integrals that gives the total perimeter of the fan blades, including both the inner and outer edges of the fan blades.

Solution: Note that $\frac{dr}{d\theta} = -3\sin(3\theta)$ and the perimeter is given by

$$\int_0^{2\pi} \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Total Perimeter = $\int_0^{2\pi} \sqrt{(\cos(3\theta) + \frac{1}{2})^2 + 9\sin^2(3\theta)} d\theta$

d. [4 points] Write, but do not evaluate, an expression giving the total area of all 3 fan blades (the shaded region of the graph). (*Hint*: Your answer from (b) may be handy, but is not strictly necessary)

Solution: Using (b), the small dashed loop is traced out for $\frac{2\pi}{9} \le \theta \le \frac{4\pi}{9}$ and the large dashed loop is traced out for $\frac{4\pi}{9} \le \theta \le \frac{8\pi}{9}$, so the area of one small loop is

$$\int_{\frac{2\pi}{9}}^{\frac{4\pi}{9}} \frac{(\cos(3\theta) + \frac{1}{2})^2}{2} d\theta$$

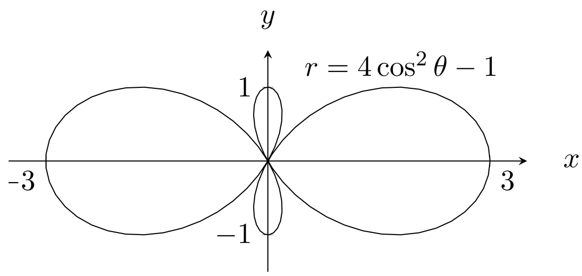
and the area of one large loop is

$$\int_{\frac{4\pi}{9}}^{\frac{8\pi}{9}} \frac{(\cos(3\theta) + \frac{1}{2})^2}{2} d\theta.$$

Exploiting the symmetry of the fan, we get the total area below.

Total Area = $3\left[\int_{\frac{4\pi}{9}}^{\frac{8\pi}{9}} \frac{(\cos(3\theta) + \frac{1}{2})^2}{2} d\theta - \int_{\frac{2\pi}{9}}^{\frac{4\pi}{9}} \frac{(\cos(3\theta) + \frac{1}{2})^2}{2} d\theta\right]$

7. [15 points] Nat is sailing a boat in a lake, with the path given by the following polar graph.



a. [4 points] What are all the angles θ, with $0 \le \theta \le 2\pi$, for which the graph passes through the origin?

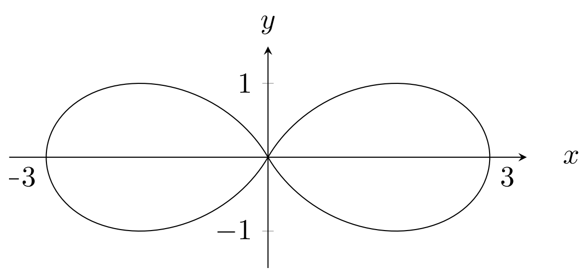
Solution: At the origin, $r = 0$.

$$4\cos^2\theta - 1 = 0$$
$$\cos^2\theta = \frac{1}{4}$$

$$\cos\theta = \frac{1}{2} \qquad \text{or} \qquad \cos\theta = -\frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3} \qquad \text{or} \qquad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}.$$

So $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

b. [4 points] Write down, but do **not** evaluate, one or more integral(s) that gives the arc length of the larger horizontal figure 8 from the path above, as given in the following graph.



Solution: When $\theta = 0$, we have $r = 4 - 1 = 3$, so the x, y-coordinates of the point is $(3, 0)$. When θ increases, the graph travels counter-clockwise. The part in the 1st quadrant is traced when θ increases from $\theta = 0$ til the first time the graph meets the origin, i.e. $\pi/3$. As for the 4th quadrant, the portion of the graph is traced when θ decreases from $\theta = 0$ til the first time the graph meets the origin at a negative angle, i.e. $\theta = 5\pi/3 - 2\pi = -\pi/3$. Therefore, the right loop has arc length

$$\int_{-\pi/3}^{\pi/3} \sqrt{(4\cos^2\theta - 1)^2 + (-8\cos\theta\sin\theta)^2}\, d\theta.$$

By symmetry, the total arc length of the horizontal figure 8 is given by doubling the right loop, i.e.

$$2\int_{-\pi/3}^{\pi/3} \sqrt{(4\cos^2\theta - 1)^2 + (-8\cos\theta\sin\theta)^2}\, d\theta.$$

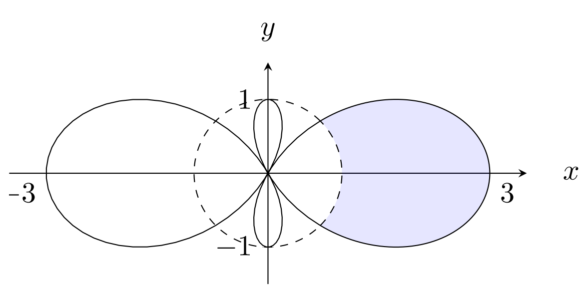
Alternatively, the left loop is traced from $\theta = 2\pi/3$ to $\theta = 4\pi/3$, so the total arc length can also be written as

$$\text{left loop} + \text{right loop} = \int_{2\pi/3}^{4\pi/3} \sqrt{(4\cos^2\theta - 1)^2 + (-8\cos\theta\sin\theta)^2}\, d\theta + \int_{-\pi/3}^{\pi/3} \sqrt{(4\cos^2\theta - 1)^2 + (-8\cos\theta\sin\theta)^2}\, d\theta.$$

Alternatively, the total arc length of the horizontal figure 8 is 4 times the arc length of the portion in the first quadrant, by symmetry, so it is

$$4\int_{0}^{\pi/3} \sqrt{(4\cos^2\theta - 1)^2 + (-8\cos\theta\sin\theta)^2}\, d\theta.$$

c. [5 points] Another boat is travelling around the unit circle $r = 1$, given by the dashed curve in the graph below. Write down, but do **not** evaluate, one or more integral(s) that gives the area of the shaded region, as shown below.



Solution: The two graphs intersect once at $0 < \theta < \pi/3$, and another time at $-\pi/3 < \theta < 0$. To solve for the angle of intersection, we set the two r to be equal.

$$4\cos^2\theta - 1 = 1$$

$$\cos^2\theta = \frac{1}{2}$$

$$\cos\theta = \frac{1}{\sqrt{2}} \qquad \text{or} \qquad \cos\theta = -\frac{1}{\sqrt{2}}$$

$$\theta = \frac{\pi}{4}, \frac{7\pi}{4} \qquad \text{or} \qquad \theta = \frac{3\pi}{4}, \frac{5\pi}{4},$$

or adding integer multiples of 2π to any of these.
We want an angle in $0 < \theta < \frac{\pi}{3}$ and another in $-\frac{\pi}{3} < \theta < 0$, so we have $\theta = \frac{\pi}{4}$ and $-\frac{\pi}{4}$.
Therefore, the area bounded by the solid curve is

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} (4\cos^2\theta - 1)^2 \, d\theta.$$

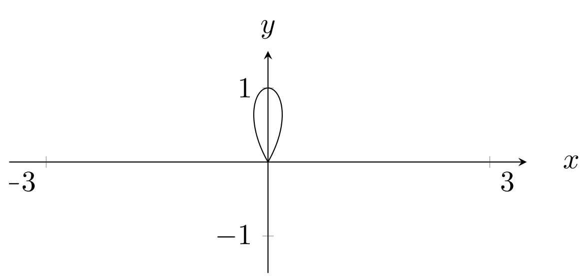
The area bounded by the dashed curve is

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} 1^2 \, d\theta = \frac{\pi}{4},$$

so the shaded area is

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} (4\cos^2\theta - 1)^2 - 1^2 \, d\theta.$$

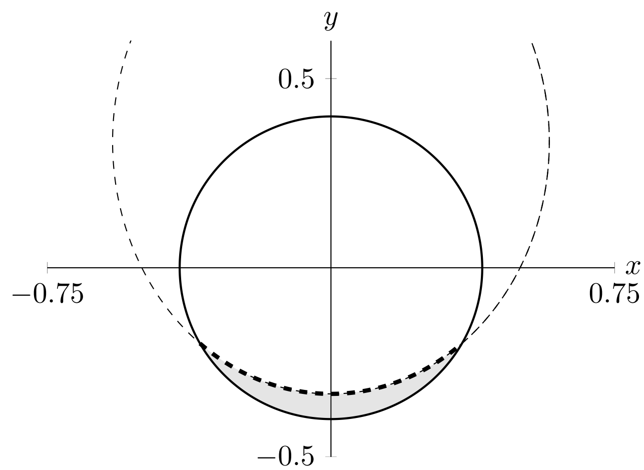
d. [2 points] Give an interval of θ-values for which the polar equation $r = 4\cos^2\theta - 1$ traces out the upper loop of the smaller figure 8 as shown below.



Solution: From the analysis in (b), we know that as θ increases from 0 to $\pi/3$, the first quadrant portion of the horizontal figure 8 is traced out.
Then as θ increases from $\pi/3$ to $2\pi/3$, we have that $r = 4\cos^2\theta - 1$ is negative. As a result, the direction of the point is **opposite** to the direction indicated by the angle θ. This means that we are below the x-axis instead of above the x-axis. As a result, the **lower** half of the small loop is traced out in $[\pi/3, 2\pi/3]$.
When θ increase from $2\pi/3$ to $4\pi/3$, the formula $r = 4\cos^2\theta - 1$ stays positive. Hence the direction is given by the angle indicated by the angle θ. At this interval, the left half of the big horizontal figure 8 is traced out.
Finally, when θ increases from $4\pi/2$ to $5\pi/3$, we have that $r = 4\cos^2\theta - 1$ is negative. Hence we need to go to the opposite direction as indicated by θ. As a result, we are at the **upper** half of the small loop when θ is in $[4\pi/3, 5\pi/3]$.

2. [9 points] Scientists are studying the bite of several different rodents. To do this, they give a wafer cookie to the animal, and take it away after the animal takes one bite.

- r is measured in inches
- The **wafer** is modeled by the region inside the polar curve
$$r = \frac{2}{5}$$
(the **solid** line in the diagram).
- The rodent's **bite** is modeled by the region inside the polar curve
$$r = \frac{1}{2 - \sin(\theta)}$$
and inside the wafer (the **dashed** line in the diagram).
- The wafer remaining after the bite is shaded in the diagram.



a. [3 points] For what values of θ between 0 and 2π does the rodent's bite meet the edge of the wafer? Justify your answer algebraically, and give your answers in **exact** form.

Solution: The bite meets the edge of the wafer when

$$\frac{2}{5} = \frac{1}{2 - \sin(\theta)},$$

which happens when $\sin(\theta) = \frac{-1}{2}$, giving us $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$.
Note that $\arcsin(-1/2) < 0$, so this does not satisfy the requirements of the problem. However, we could use this to find that $\pi - \arcsin(-1/2)$ and $2\pi + \arcsin(-1/2)$ are both between 0 and 2π.

Answer: $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

b. [3 points] Write, but do not evaluate, an expression involving one or more integrals that gives the area, in square inches, of the wafer remaining after the bite.

Answer: $\frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{4}{25} - \frac{1}{(2-\sin(\theta))^2}\,d\theta$

c. [3 points] The bite mark in the wafer is represented by the **thick** dashed line in the diagram. Write, but do not evaluate, an expression involving one or more integrals that gives the length, in inches, of this bite mark.

Solution: We use the formula for arc length and

$$\frac{dr}{d\theta} = \frac{\cos(\theta)}{(2-\sin(\theta))^2}.$$

Answer: $\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \sqrt{\frac{1}{(2-\sin(\theta))^2} + \frac{\cos^2(\theta)}{(2-\sin(\theta))^4}}\,d\theta$

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