Math 116 — Practice for Exam 3

Generated December 5, 2017

Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 5 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2013	2	8	zombies	15	
Fall 2015	2	7	currency	18	
Fall 2014	2	1	robot uprising	12	
Fall 2012	2	3	cell growth	15	
Winter 2012	2	4	tank/bank	13	
Total				73	

Recommended time (based on points): 66 minutes

8. [15 points] Two zombies are chasing Jake down the Diag. Let $J(t)$ be Jake's position, measured in meters along the Diag, as he runs from the zombies. In this problem the time t is measured in seconds.

a. [3 points] The velocity of the first zombie is proportional to the difference between its own position, $S(t)$, and Jake's position, with constant of proportionality k. Using this fact, write the differential equation satisfied by $S(t)$.

Solution:

$$\frac{dS}{dt} = k(S - J(t)).$$

b. [2 points] State whether your equation in part (a) is separable. Circle the correct answer.

Solution:

The equation is: separable **NOT SEPARABLE**

Note: $J(t)$ is not constant, since Jake is running.

c. [9 points] The position of the second zombie at time t is given by the function $Z(t)$ (in meters), and satisfies the differential equation

$$\frac{dZ}{dt} = \alpha \frac{J(t)}{Z},$$

where α is a positive constant. Assuming that $Z(0) = 5$ and that Jake's position is given by $J(t) = 2t + 10$, find a formula for $Z(t)$.

Solution: Separating gives:

$$\int Z dZ = \alpha \int 2t + 10 \; dt,$$

and so

$$\tfrac{1}{2}Z^2 = \alpha(t^2 + 10t) + C.$$

Plugging in $Z(0) = 5$, we see that $C = \frac{25}{2}$, so $Z(t)$ is given by:

$$Z(t) = \sqrt{2\alpha t^2 + 20\alpha t + 25}.$$

d. [1 point] In the differential equation $\dfrac{dZ}{dt} = \alpha \dfrac{J(t)}{Z}$, what are the units of α?

Solution: The units are m/s.

7. [18 points] A certain small country called Merrimead has 25 million dollars in paper currency in circulation, and each day 50 thousand dollars comes into Merrimead's banks. The government decides to introduce new currency by having the banks replace the old bills with new ones whenever old currency comes into the banks. Assume that the new bills are equally distributed throughout all paper currency. Let $M = M(t)$ denote the amount of new currency, in thousands of dollars, in circulation at time t days after starting to replace the paper currency.

a. [5 points] Write a differential equation involving $M(t)$, including an appropriate initial condition.

Solution: The concentration of old bills among all bills in circulation is $\frac{25000-M}{25000}$, and 50 thousand dollars moves through the bank each day, so

$$\frac{dM}{dt} = \text{Concentration} \times \text{Money per day} = \frac{25000 - M}{25000} \cdot 50, \qquad M(0) = 0.$$

Now consider the differential equation

$$B^2 + 2B\frac{dB}{dt} = 2500.$$

b. [4 points] Find all equilibrium solutions and classify their stability.

Solution: If $\frac{dB}{dt} = 0$ then we see that $B^2 = 2500$, so the equilibrium solutions are $B = \pm 50$. Both equilibrium solutions are stable.

Brightcrest, a second small country, also wants to replace all of their old paper bills as well, using a different strategy than Merrimead. The amount $B(t)$, in millions of dollars, of new paper currency in circulation in Brightcrest at a time t years after starting to replace the paper currency is modeled by the differential equation for B above with initial condition $B(0) = 0$.

c. [6 points] Find a formula for $B(t)$.

Solution: Using separation of variables, we have $\int \frac{2B}{2500-B^2}\, dB = \int dt$. Using the substitution $w = 2500 - B^2, dw = -2B\, dB$, we see that $-\ln|2500 - B^2| = t + C$. Solving for B, we see $B = \sqrt{2500 - Ae^{-t}}$.
Since $B(0) = 0$, we see that $A = 2500$, so

$$B(t) = \sqrt{2500 - 2500e^{-t}}.$$

d. [3 points] Assuming that all of the old bills are replaced in the long run, how much time will pass after starting to replace the paper bills until the new currency accounts for 99% of all currency in Brightcrest?

Solution: Since all of the bills are replaced in the long run, the total amount of money in circulation is $\lim_{t\to\infty} B(t) = 50$ million dollars. So the amount of time that passes until the new bills account for 99% of all currency is the value of t so that

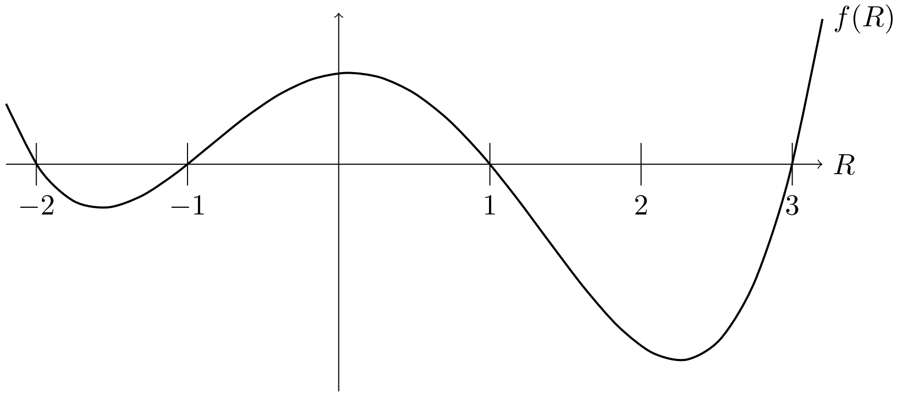
$$(.99)(50) = \sqrt{2500 - 2500e^{-t}}.$$

So $t = -\ln(1 - (.99)^2) \approx 3.92$ years.

1. [12 points] Franklin, your robot, is on the local news. Let $R(t)$ be the number of robots, in millions, that have joined the robot uprising t minutes after the start of the broadcast. After watching the news for a little bit, you find that $R(t)$ obeys the differential equation:

$$\frac{dR}{dt} = f(R)$$

for some function $f(R)$. A graph of $f(R)$ is shown below.



a. [3 points] If $R(t)$ is the solution to the above differential equation with $R(0) = 0$, what is $\lim_{t\to\infty} R(t)$? Justify your answer.

Solution: If $R = 0$, $f(R) = R'(t)$ is positive, so R will increase as t increases. As R increases to 1, $R'(t) = f(R)$ goes to 0, so $\lim_{t\to\infty} R(t) = 1$.

b. [6 points] Find the equilibrium solutions to the above differential equation **and** classify them as stable or unstable.

Solution:

$R = -2$	$\boxed{\textbf{Stable}}$	**Unstable**
$R = -1$	**Stable**	$\boxed{\textbf{Unstable}}$
$R = 1$	$\boxed{\textbf{Stable}}$	**Unstable**
$R = 3$	**Stable**	$\boxed{\textbf{Unstable}}$

c. [3 points] Let $R(t)$ be a solution to the given differential equation, with $R(3) = 0.5$. Is the graph of $R(t)$ concave up, concave down, or neither at the point $(3, 0.5)$? Justify your answer.

Solution:

$$\frac{d^2R}{dt^2} = \frac{d}{dt}f(R) = f'(R)\frac{dR}{dt} = f'(R)f(R)$$

At $R = 0.5$, $f'(0.5) < 0$ and $f(0.5) > 0$ so $\frac{d^2R}{dt^2} < 0$. Therefore, the solution curve will be concave down.

3. [15 points] A model for cell growth states that the volume $V(t)$ (in mm^3) of a cell at time t (in days) satisfies the differential equation

$$\frac{dV}{dt} = 2e^{-t}V.$$

a. [2 points] Find the equilibrium solutions of this equation.

Solution: $V = 0$.

b. [8 points] Solve the differential equation. The initial volume of the cell is V_0 mm^3. Your answer should contain V_0.

Solution:

$$\begin{aligned}
\frac{dV}{dt} &= 2e^{-t}V \\
\frac{dV}{V} &= 2e^{-t}dt \\
\ln|V| &= -2e^{-t} + C \\
V &= Be^{-2e^{-t}}. \\
V_0 &= Be^{-2} \\
B &= V_0e^2 \\
V &= V_0e^2e^{-2e^{-t}} = V_0e^{2-2e^{-t}}.
\end{aligned}$$

c. [3 points] How long does it take a cell to double its initial size?

Solution:

$$\begin{aligned}
2V_0 &= V_0e^{2-2e^{-t}} \\
2 &= e^{2-2e^{-t}} \\
\ln 2 &= 2 - 2e^{-t} \\
2e^{-t} &= 2 - \ln 2 \\
e^{-t} &= \frac{2 - \ln 2}{2} \\
t &= -\ln\left(\frac{2 - \ln 2}{2}\right).
\end{aligned}$$

d. [2 points] What happens to the value of the volume of the cell in the long run?

Solution: $\lim_{t\to\infty} V(t) = \lim_{t\to\infty} V_0e^{2-2e^{-t}} = V_0e^2$. Hence the volume of the cell $V(t)$ approaches the value V_0e^2.

4. [13 points]

a. [6 points] A cylindrical tank with height 8 m and radius of 8 m is standing on one of its circular ends. The tank is initially empty. Water is added at a rate of 2 m^3 /min. A valve at the bottom of the tank releases water at a rate proportional to the water's depth (proportionality constant $= k$). Let $V(t)$ be the volume of the water in the tank at time t, and $h(t)$ be the depth of the water at time t.

i. Find a formula for $V(t)$ in terms of $h(t)$. $V(t) =$ ____________________

ii. Find the differential equation satisfied by $V(t)$. Include initial conditions.

Solution: i) The formula is $V(t) = 64\pi h(t)$.

ii) The differential equation is

$$\frac{dV}{dt} = 2 - kh.$$

So now we can solve $h(t) = \frac{V(t)}{64\pi}$. Substituting in V for h, we get

$$\frac{dV}{dt} = 2 - k\frac{V}{64\pi}$$

with initial condition $V(0) = 0$.

b. [7 points] Let $M(t)$ be the balance in dollars in a bank account t years after the initial deposit. The function $M(t)$ satisfies the differential equation

$$\frac{dM}{dt} = \frac{1}{100}M - a.$$

where a is a positive constant. Find a formula for $M(t)$ if the initial deposit is $1,000$ dollars. Your answer may depend on a.

Solution: This equation is separable:

$$\frac{dM}{M - 100a} = \frac{1}{100}dt.$$

Integrating, we find $\ln|M - 100a| = \frac{t}{100} + C$. So we get

$$M = Be^{t/100} + 100a.$$

Using the initial conditions, $M(0) = 1000$, so $1000 = B + 100a$. Substituting back in we get

$$M = 100\left((10 - a)e^{t/100} + a\right).$$