Math 116 — Practice for Exam 2

Generated October 1, 2017

Name: **SOLUTIONS**

Instructor: ______________________ Section Number: __________

1. This exam has 5 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2014	3	7	chickens vs. robots	14	
Fall 2015	1	8	boxers2	13	
Fall 2016	2	10	tracking chip	14	
Winter 2017	2	6	Venice Beach	13	
Winter 2015	2	1	spinning	16	
Total				70	

Recommended time (based on points): 67 minutes

7. [14 points] Chickens continue to appear around you, and Franklin's army is hesitant to advance.

a. [6 points] Let $F(t)$ give the total number of chickens that have arrived after t seconds. You observe that $F(t)$ obeys the following differential equation

$$\frac{dF}{dt} = e^{-F}t^2.$$

If there are initially 20 chickens, find a formula (in terms of t) for $F(t)$.

Solution:

$$\int e^F df = \int t^2 dt$$

$$e^F = \frac{t^3}{3} + C$$

$$F(t) = \ln(\frac{t^3}{3} + C)$$

Since $F(0) = 20$, we see that

$$20 = \ln(C)$$

so $C = e^{20}$, and

$$F(t) = \ln(\frac{t^3}{3} + e^{20})$$

b. [4 points] A large, familiar-looking chicken steps forward from the flock and clucks, "Koo Koo Katcha!". This large chicken waddles towards Franklin following the parametric equations

$$x(t) = \frac{\sin(\pi t) + 1}{\pi} \qquad y(t) = \ln(t + 1)$$

where t is the time, in seconds, after the chicken steps forward from the flock and both x and y are measured in feet. Find the chicken's speed 10 seconds after it steps forward. Include units.

Solution:

$$x'(t) = \cos(\pi t) \qquad y'(t) = \frac{1}{t+1}$$

Now we plug these into the speed formula

$$\text{Speed} = \sqrt{(x'(t))^2 + (y'(t))^2)}$$

when $t = 10$.

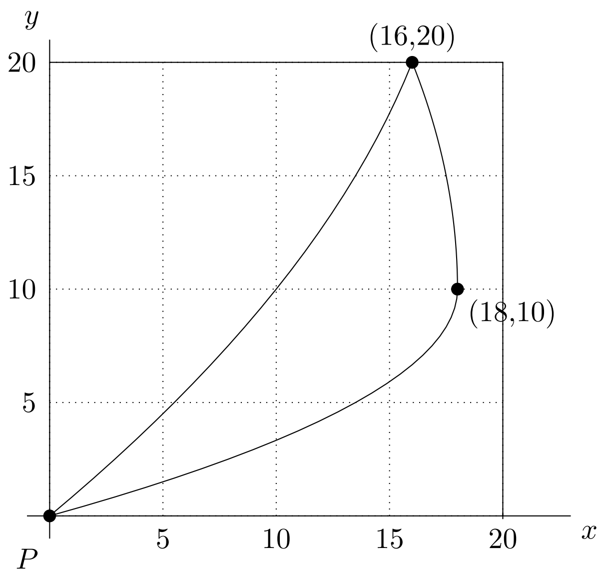
$$\text{Speed} = \sqrt{(\cos(10\pi))^2 + (\frac{1}{11})^2} = \frac{\sqrt{122}}{11}$$

c. [4 points] Franklin says, "`BEEP BOOP BEEP. YOU'RE RIGHT, WHAT HAVE I BECOME?`" A single robot tear falls from Franklin's robot eye. Consider the region in the xy-plane bounded by $y = \dfrac{\sin(x)}{x+2}$, $x = \pi$, $x = 2\pi$, and the x-axis. The volume of Franklin's tear is given by rotating this region around the x-axis. Write an integral giving the volume of Franklin's tear. Do not evaluate this integral.

Solution:

$$\int_{\pi}^{2\pi} \pi \left(\frac{\sin(x)}{x+2}\right)^2 dx$$

8. [13 points] During the first round of the rematch between Paul "Stretch" Cassenick and Stephen "Dee" Boxer, Paul's position in the boxing ring t minutes after the 3-minute round began is given by $(x(t), y(t))$ where $x(t)$ and $y(t)$ are Paul's distance from his corner, in feet, in the $x-$ and $y-$directions, respectively. The ring is the 20x20 foot square pictured below, and the point P is Paul's corner. Suppose $x(t) = -8t(t-3)$, and $y(t)$ has values given in the table below and is **linear** between each consecutive pair of t-values in the table.



t	$y(t)$
0	0
1	20
1.5	10
3	0

a. [5 points] On the diagram of the ring, sketch a graph of Paul's path through the ring during the first round of the rematch. Label the points corresponding to Paul's position at $t = 1$ and $t = 1.5$ with their x- and y-coordinates.

b. [4 points] Find the slope of the tangent line to Paul's path at $t = 2$.

Solution: $$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\big|_{t=2}\right)}{\left(\frac{dx}{dt}\big|_{t=2}\right)} = \frac{\frac{-20}{3}}{-8} = \frac{5}{6}.$$

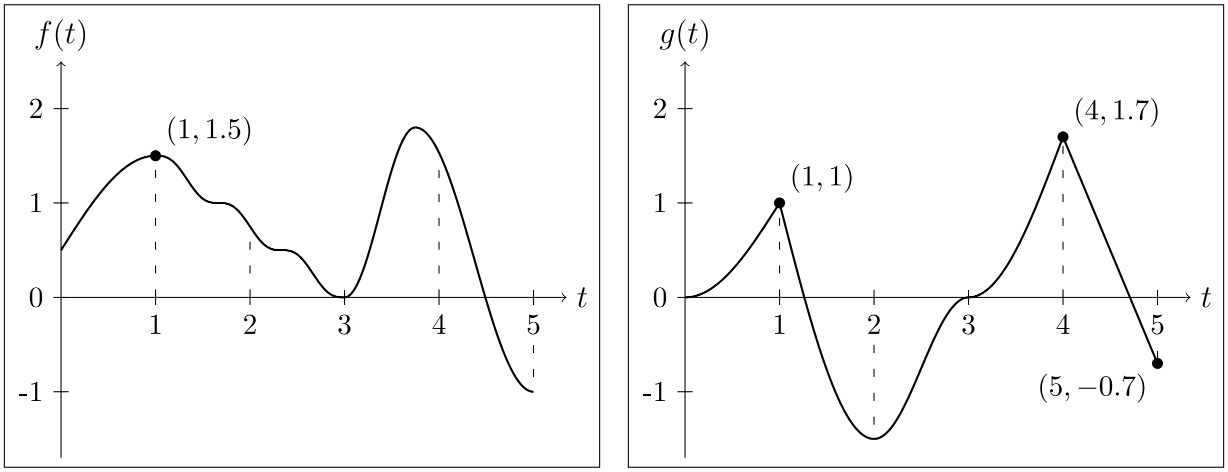
c. [4 points] Write an explicit expression involving integrals that gives the distance Paul traveled during the first minute of the round. Your answer should not contain the letters 'x' or 'y'.

Solution: The distance traveled over the first minute is the arc length of the curve from $t = 0$ to $t = 1$. So the distance Paul traveled is $\int_0^1 \sqrt{(24-16t)^2 + (20)^2}dt$ feet.

10. [14 points] Fearing that she is losing authority over her robot ward, Dr. Durant has installed a tracking chip in Steph's mainframe. The chip gives Steph's location separately in x- and y-coordinates, where the units of the axes are miles, Dr. Durant's office corresponds to the origin $(x, y) = (0, 0)$, the positive y-axis points north, and the positive x-axis points east. On night 1, Dr. Durant noticed unusual levels of activity; t hours after midnight, Steph began moving according to the parametric equations

$$x = f(t) \qquad\qquad y = g(t),$$

where $f(t)$ and $g(t)$ are plotted below for $0 \leq t \leq 5$.



a. [2 points] When was Steph farthest north and south on night 1? Write your answers in the blanks provided. You do **not** need to show your work.

Solution: **North:** ___4___ **a.m.** **South:** ___2___ **a.m.**

b. [3 points] What was Steph's speed at $t = 4.9$ on night 1? You may use the fact that $f'(4.9) = -1$. Include units.

Solution: Her speed is $\sqrt{(-1)^2 + (-2.4)^2} = \sqrt{6.76}$ mi/hr.

c. [2 points] What direction was Steph moving at $t = 2$ on night 1? Circle only one answer.

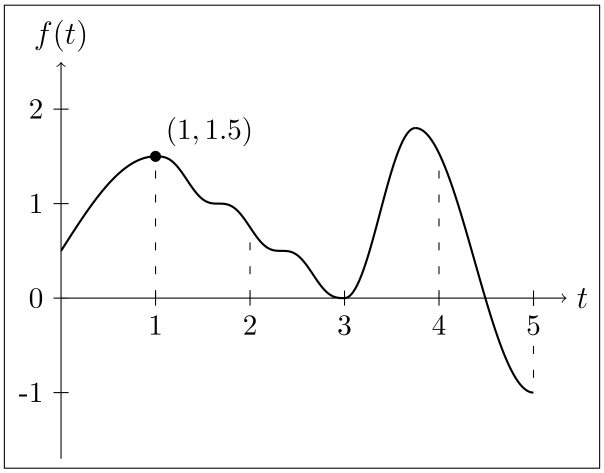
NORTH AND EAST EAST ONLY SOUTH AND EAST

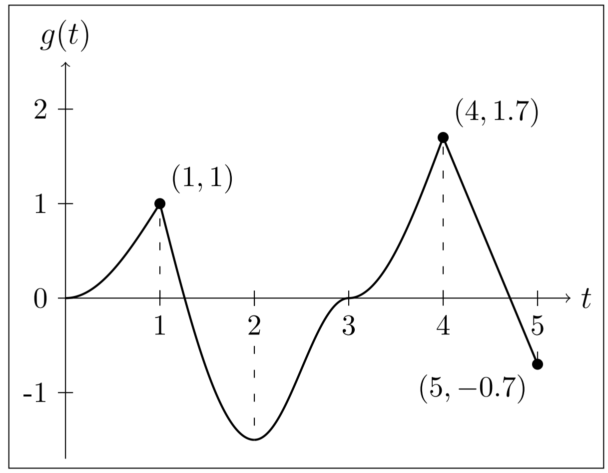
NORTH AND WEST **[WEST ONLY]** SOUTH AND WEST

10 (continued). Recall that on night 1, Steph's position was given by the parametric equations

$$x = f(t) \qquad\qquad y = g(t),$$

where $f(t)$ and $g(t)$ are plotted below for $0 \le t \le 5$. As before, Dr. Durant's office is at the origin $(x, y) = (0, 0)$, the positive y-axis points north, and the positive x-axis points east.





d. [3 points] How far away was Steph from Dr. Durant's office at $t = 1$ on night 1?

Solution: Steph was $\sqrt{1^2 + 1.5^2} = \sqrt{3.25}$ mi away.

On night 2, Steph's movements were even stranger, following the parametric equations

$$x = \int_0^t f(s)\,ds \qquad\qquad y = \int_0^t g(s)\,ds.$$

e. [2 points] What direction was Steph moving at $t = 2$ on night 2? Circle only one answer.

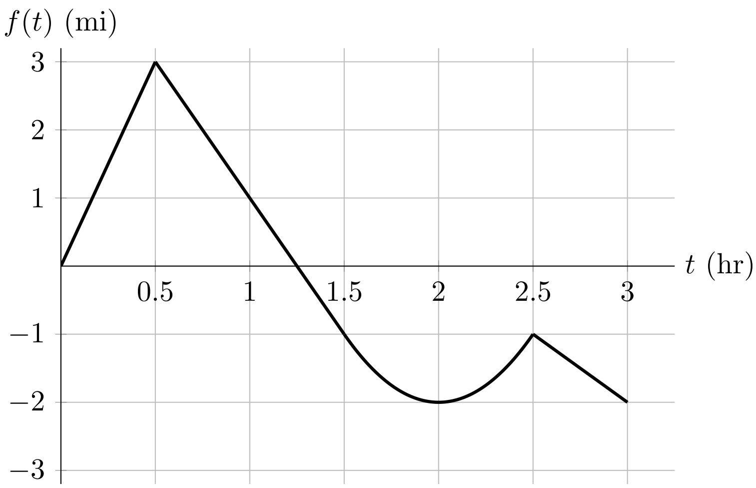
NORTH AND EAST **EAST ONLY** $\boxed{\textbf{SOUTH AND EAST}}$

NORTH AND WEST **WEST ONLY** **SOUTH AND WEST**

f. [2 points] Did Steph come to a stop between midnight and 5 a.m. on night 2? If so, at what time(s) did she come to a stop?

Solution: Yes; she came to a stop at 3 a.m.

6. [13 points] Anderson and Glen decide to take a road trip starting from Venice Beach. They have no worries about getting anywhere quickly, as they enjoy each other's company, so they take a very inefficient route. Suppose that Venice Beach is located at $(0,0)$ and that Anderson and Glen's position (x, y) (measured in miles) t hours after leaving Venice Beach is given by a pair of parametric equations $x = f(t)$, $y = g(t)$. A graph of $f(t)$ and a formula for $g(t)$ are given below. Note that $f(t)$ is linear on the intervals $[0, 0.5]$, $[0.5, 1.5]$, and $[2.5, 3]$.



$$g(t) = -t^3 + 5t^2 - 3t$$

Note: For each of the following, your final answer should **not** involve the letters f and g.

a. [2 points] If their roadtrip last 3 hours, what are the $x-$ and $y-$ coordinates of their final destination?

Solution: Note that at time $t = 3$, we have $x = f(3) = -2$ and $y = g(3) = 9$. So the coordinates of their final destination are $(-2, 9)$.

b. [3 points] At what speed are they traveling 2 hours into their trip?

Solution: We have $\left.\frac{dx}{dt}\right|_{t=2} = f'(2) = 0$ and $\left.\frac{dy}{dt}\right|_{t=2} = g'(2) = 5.$
So their speed at time $t = 2$ is $\sqrt{0^2 + 5^2} = 5$ miles per hour.

c. [4 points] Write, but do not compute, an expression involving one or more integrals that gives the distance they traveled, in miles, in the first **half** hour of their trip.

Solution: On the interval $(0, 0.5)$, we see that $f(t) = 6t$, so on this interval, we have

$$f'(t) = 6 \quad \text{and} \quad g'(t) = -3t^2 + 10t - 3.$$

The parametric arc length formula then implies that the distance they travelled from $t = 0$ to $t = 0.5$ is $\int_0^{0.5} \sqrt{(6)^2 + (-3t^2 + 10t - 3)^2}\, dt$ miles.

d. [4 points] Write down a pair of parametric equations using the parameter s for the line tangent to their path at $t = 2.75$ hours.

Solution: Note that

$$f(2.75) = -1.5, \quad \left.\frac{df}{dt}\right|_{t=2.75} = -2, \quad g(2.75) = 8.765625, \quad \text{and} \quad \left.\frac{dg}{dt}\right|_{t=2.75} = 1.8125$$

There are many possible parametrizations. There is no need to have this match with the parameter t from earlier, so the answer below has the line passing through $(-1.5, 8.765625)$ at $s = 0$.

Answer: $x(s) =$ $-2s - 1.5$ and $y(s) =$ $1.8125s + 8.765625$

1. [16 points] Carla and Bobby run a race after spinning in circles for a good amount of time to make themselves dizzy. They start at the origin in the xy-plane and they race to the line $y = 5$. Assume the units of x and y are meters.
Bobby's position in the xy-plane t seconds after the races starts is

$$(-\sqrt{3}t\cos t, \frac{1}{\sqrt{3}}t\sin t)$$

and Carla's position in the xy-plane t seconds after the race starts is

$$(t\sin t, -t\cos t).$$

a. [4 points] Write an integral that gives the distance that Carla travels during the first two seconds of the race. Do not evaluate your integral.

Solution: We have

$$\frac{dx}{dt} = \sin(t) + t\cos(t),$$
$$\frac{dy}{dt} = -\cos(t) + t\sin(t).$$

The distance traveled by Carla in the first two seconds of the race is then given by $\int_0^2 \sqrt{(\sin(t) + t\cos(t))^2 + (t\sin(t) - \cos(t))^2}dt.$

b. [3 points] Find Carla's speed at $t = \pi$.

Solution: We have that Carla's speed is given by the function $\sqrt{(\sin(t) + t\cos(t))^2 + (t\sin(t) - \cos(t))^2}$, and so we need only plug in $t = \pi$ which gives us the value below.

Carla's speed at $t = \pi$ is $\sqrt{\pi^2 + 1}$ **m/sec**

c. [4 points] Carla and Bobby are so dizzy that they run into each other at least once during the race. Find the first time $t > 0$ that they run into each other, and give the point (x, y) where the collision occurs.

Solution: Setting the x and y coordinate functions equal gives us that collisions will occur when $\tan(t) = -\sqrt{3}$. The first time for $t > 0$ when this occurs is $2\pi/3$. Plugging this t value into either the equations for Bobby's or Carla's position will give the (x, y) coordinates given below for where the collision occurs.

They first run into each other at $t =$ $\frac{2\pi}{3}$

The collision occurs at $(x, y) =$ $(\frac{\sqrt{3}}{3}\pi, \frac{\pi}{3})$

d. [5 points] Bobby's phone flies out of his pocket at $t = \pi/2$. It travels in a straight line in the same direction as he was moving at $t = \pi/2$. Find the equation of this line in Cartesian coordinates.

Solution: Plug in $t = \frac{\pi}{2}$ to the parametric equations for Bobby's position to get that Bobby is at the point $P = \left(0, \frac{\pi}{2\sqrt{3}}\right)$ at $t = \frac{\pi}{2}$. We can find the slope of the curve at that point

$$\left.\frac{dy}{dx}\right|_P = \frac{\frac{dy}{dt}|_{t=\pi}}{\frac{dx}{dt}|_{t=\pi}} = \frac{\frac{1}{\sqrt{3}}\sin(\frac{\pi}{2}) + \frac{\pi}{2\sqrt{3}}\cos(\frac{\pi}{2})}{-\sqrt{3}\cos(\frac{\pi}{2}) + \frac{\sqrt{3}\pi}{2}\sin(\frac{\pi}{2})} = \frac{2}{3\pi}.$$

Since the line that gives the path of the phone is the same as the tangent line to the curve giving Bobby's motion at the point P, the equation of the line we want is that given below.

The equation for the line is $y = \frac{2}{3\pi}x + \frac{\pi}{2\sqrt{3}}$