MATH 116 — PRACTICE FOR EXAM 2

Generated November 6, 2017

NAME: SOLUTIONS

INSTRUCTOR: ____________________ SECTION NUMBER: ________

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1. This exam has 14 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a 3″ × 5″ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2014	3	8		12	
Fall 2014	3	6		8	
Fall 2015	3	13		10	
Fall 2016	3	1		4	
Winter 2012	3	4 (c)		4 [illegible]	
~~Fall 2012~~	~~3~~	~~2~~		~~[illegible]~~	
Winter 2013	2	4 (a)		8 [illegible]	
Fall 2013	2	6 (c)		4 [illegible]	
Winter 2014	2	10		12	
Fall 2014	2	7		8	
Winter 2015	2	10		15	
Winter 2016	3	10		14	
Fall 2016	2	9		10	
Winter 2017	2	5		10	
Total				119 [illegible]	

8. [12 points] Suppose a_n and b_n are sequences of positive numbers with the following properties.

- $\sum_{n=1}^{\infty} a_n$ converges.
- $\sum_{n=1}^{\infty} b_n$ diverges.
- $0 < b_n \leq M$ for some positive number M.

For each of the following questions, circle the correct answer. No justification is necessary.

a. [2 points] Does the series $\sum_{n=1}^{\infty} a_n b_n$ converge?

Since $0 < a_n b_n \leq M a_n$ and $\sum_{n=1}^{\infty} M a_n$ converges, $\sum_{n=1}^{\infty} a_n b_n$ converges by Comparison Test

[✓ Converge] Diverge Cannot determine

b. [2 points] Does the series $\sum_{n=1}^{\infty} (-1)^n b_n$ converge?

Diverges if $b_n = 1$ for every n (by n^{th}-term test); Converges if $b_n = \frac{1}{n}$ (by AST).

Converge Diverge [✓ Cannot determine]

c. [2 points] Does the series $\sum_{n=1}^{\infty} \sqrt{b_n}$ converge?

This must diverge - if it converged then n^{th} term test implies $\lim_{n\to\infty} \sqrt{b_n} = 0$, so that in particular $0 < \sqrt{b_n} < 1$ for all big n, whence $0 < b_n < \sqrt{b_n}$ for all big n, so that $\sum_{n=1}^{\infty} b_n$ converges, contradicting the hypothesis of the problem.

Converge [✓ Diverge] Cannot determine

d. [2 points] Does the series $\sum_{n=1}^{\infty} \sin(a_n)$ converge?

Since $\sum_{n=1}^{\infty} a_n$ converges, n^{th} term test implies $\lim_{n\to\infty} a_n = 0$, so that $0 < a_n < \pi$ for all big n. Also $\lim_{n\to\infty} \frac{\sin(a_n)}{a_n} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1$, so LCT implies that $\sum_{n=1}^{\infty} \sin(a_n)$ converges because $\sum_{n=1}^{\infty} a_n$ converges.

[✓ Converge] Diverge Cannot determine

e. [2 points] Does the series $\sum_{n=1}^{\infty} (a_n + b_n)^2$ converge?

Converges if $a_n = 0$ and $b_n = \frac{1}{n}$ (by p-test). Diverges if $a_n = 0$ and $b_n = \frac{1}{\sqrt{n}}$ (by p-test).

Converge Diverge [✓ Cannot determine]

f. [2 points] Does the series $\sum_{n=1}^{\infty} e^{-b_n}$ converge?

Diverges by n^{th} term test, since $e^{-M} \leq e^{-b_n}$ implies that $\lim_{n\to\infty} e^{-b_n} \neq 0$

Converge [✓ Diverge] Cannot determine

6. [8 points] Suppose that $f(x)$, $g(x)$, $h(x)$ and $k(x)$ are all positive, differentiable functions. Suppose that

$$0 < f(x) < \frac{1}{x} < g(x) < \frac{1}{x^2}$$

for all $0 < x < 1$, and that

$$0 < h(x) < \frac{1}{x^2} < k(x) < \frac{1}{x}$$

for $x > 1$. Determine whether the following statements are always, sometimes or never true by circling the appropriate answer. No justification is necessary.

a. [2 points] $\int_0^1 g(x)dx$ converges.

[Diverges by Comparison Test, since $0<\frac{1}{x}<g(x)$ and $\int_0^1 \frac{1}{x}dx=\infty$.]

Always **Sometimes** [✓ **Never**]

b. [2 points] $\int_0^1 f(x)dx$ diverges.

[Converges if $f(x)$ is very close to zero;
diverges if $f(x)$ is very close to $\frac{1}{x}$.
Explicit examples: conv. if $f(x)=x$
div. if $f(x)=\frac{1}{x}-x$.]

Always [✓ **Sometimes**] **Never**

c. [2 points] $\sum_{n=1}^{\infty} h(n)$ diverges.

[Converges by Comparison Test, since $0<h(x)<\frac{1}{x^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ conv. by p-test ($p=2$).]

Always **Sometimes** [✓ **Never**]

d. [2 points] $\sum_{n=1}^{\infty} k(n)$ converges.

[Converges if $k(x)$ is very close to $\frac{1}{x^2}$;
diverges if $k(x)$ is very close to $\frac{1}{x}$.
Explicit examples: conv. if $k(x)=\frac{1}{x^2}+\frac{1}{x^3}$ for all big enough x,
div. if $k(x)=\frac{1}{x}-\frac{1}{x^3}$ for all big enough x.]

Always [✓ **Sometimes**] **Never**

13. [10 points] Suppose a_n and b_n are sequences with the following properties.

- $\sum_{n=1}^{\infty} a_n$ converges.
- $n \le b_n \le e^n$.

For each of the following statements, decide whether the statement is always true, sometimes true, or never true. Circle your answer. No justification is necessary. **You only need to answer 5 of the 7 questions.** Only answer the 5 questions you want graded. If it is unclear which 5 questions are being answered, the first 5 questions you answer will be graded.

a. [2 points] The sequence $\frac{1}{b_n}$ diverges.

[Converges to 0, since $\frac{1}{e^n} \le \frac{1}{b_n} \le \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{e^n} = 0 = \lim_{n\to\infty} \frac{1}{n}$. "Squeeze theorem"] from Math 115

ALWAYS SOMETIMES [✓ NEVER]

b. [2 points] The sequence a_n is bounded.

[Since $\sum_{n=1}^{\infty} a_n$ converges, n^{th} term test $\Rightarrow \lim_{n\to\infty} a_n = 0$, so that $-0.1 < a_n < 0.1$ for all big enough n. But there are only finitely many n's which aren't "big enough", and any finite set of numbers is bounded, so the a_n's are bounded.]

[✓ ALWAYS] SOMETIMES NEVER

c. [2 points] The series $\sum_{n=1}^{\infty} \frac{1}{b_n}$ diverges.

[Converges if $b_n = e^n$ (geometric series with ratio $\frac{1}{e}$).
Diverges if $b_n = n$ (p-test, $p=1$).]

ALWAYS [✓ SOMETIMES] NEVER

d. [2 points] The series $\sum_{n=1}^{\infty} e^{-a_n}$ converges.

[Since $\sum_{n=1}^{\infty} a_n$ converges, p-test $\Rightarrow \lim_{n\to\infty} a_n = 0$, so $\lim_{n\to\infty} e^{-a_n} = e^{-\lim_{n\to\infty} a_n} = e^{-0} = 1 \neq 0$, whence $\sum_{n=1}^{\infty} e^{-a_n}$ diverges by n^{th} term test.]

ALWAYS SOMETIMES [✓ NEVER]

e. [2 points] The series $\sum_{n=1}^{\infty} a_n^2$ diverges.

[Converges if $a_n = 0$.
Diverges if $a_n = \frac{(-1)^n}{\sqrt{n}}$, since $\sum_{n=1}^{\infty} a_n$ converges by AST and $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-test ($p=1$).]

ALWAYS [✓ SOMETIMES] NEVER

f. [2 points] The series $\sum_{n=1}^{\infty} a_n b_n$ converges.

[Converges if $a_n = 0$.
Diverges if $a_n = \frac{1}{n^2}$ and $b_n = e^n$, by n^{th} term test, since $\lim_{n\to\infty} \frac{e^n}{n^2} = \infty \neq 0$.]

ALWAYS [✓ SOMETIMES] NEVER

g. [2 points] The series $\sum_{n=1}^{\infty} \frac{b_n}{n!}$ converges.

[Converges by Comparison Test since $0 < \frac{b_n}{n!} \le \frac{e^n}{n!}$ and $\sum_{n=1}^{\infty} \frac{e^n}{n!}$ converges by Ratio Test, as

$$\lim_{n\to\infty} \left| \frac{\frac{e^{n+1}}{(n+1)!}}{\frac{e^n}{n!}} \right| = \lim_{n\to\infty} \frac{e}{n+1} = 0.$$

[✓ ALWAYS] SOMETIMES NEVER

1. [4 points] Suppose that the power series $\sum_{n=0}^{\infty} c_n(x-3)^n$ converges at $x = 6$ and diverges at $x = -2$. What can you say about the behavior of the power series at the following values of x? For each part, circle the correct answer. Ambiguous responses will be marked incorrect.

a. [1 point] At $x = -3$, the power series...

CONVERGES [✓ **DIVERGES**] **CANNOT DETERMINE**

b. [1 point] At $x = 0$, the power series...

CONVERGES **DIVERGES** [✓ **CANNOT DETERMINE**]

c. [1 point] At $x = 8$, the power series...

CONVERGES **DIVERGES** [✓ **CANNOT DETERMINE**]

d. [1 point] At $x = 2$, the power series...

[✓ **CONVERGES**] **DIVERGES** **CANNOT DETERMINE**

Examples:
conv. if $c_n = \frac{1}{4^n}$
div. if $c_n = \frac{1}{n \cdot 3^n}$.

Examples:
conv. if $c_n = \frac{1}{n \cdot (-5)^n}$
div. if $c_n = \frac{1}{4^n}$.

~~**2.** [5 points] Determine the **radius** of convergence of the power series~~

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} x^{2n}.$$

~~Justify your work carefully and write your final answer in the space provided. Limit syntax will be enforced.~~

~~*Solution:* For $n = 0, 1, \ldots,$ let $a_n = \frac{(2n)!}{(n!)^2}$. We have~~

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(2(n+1))!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!} = \frac{(2n+1)(2n+2)}{(n+1)^2} \to 4$$

~~as $n \to \infty$. Hence the radius of convergence is $\sqrt{\frac{1}{4}} = \frac{1}{2}$.~~

For this problem, the interval of convergence is centered at 3.

Since the series converges at $x=6$, [illegible] the radius of convergence is at least $|6-3|=3$.

Since the series [illegible] diverges at $x=-2$, the radius of convergence is at most $|-2-3|=5$.

Say the radius of convergence is R.

~~Radius of convergence = $\frac{1}{2}$~~

Then the series converges at x if $|x-3|<R$, and diverges at x if $|x-3|>R$, but there is no general rule about what happens when $|x-3|=R$.

Thus all we know is that the series converges if $x=6$ or $|x-3|<|6-3|$, i.e., it converges if $0<x\le 6$; and the series diverges if $x=-2$ or $|x-3|>|-2-3|$.

4, **c.** [4 points] Suppose $h(x)$ and $f(x)$ are continuous functions satisfying

i. $0 < f(x) \leq \frac{1}{x^p}$ for $0 < x \leq 1$.

ii. $\frac{1}{x^{p+\frac{1}{2}}} \leq h(x) \leq \frac{1}{x^p}$ for $x \geq 1$.

Decide whether each of the following expressions converge, diverge or if there is not enough information available to conclude.

Solution:

i. If $p = \frac{1}{2}$, [Since $p=\frac{1}{2}$, $\frac{1}{x} \leq h(x) \leq \frac{1}{x^{1/2}}$.]

(a) $\lim_{x\to\infty} h(x)$ [Converges to 0, since $\lim_{x\to\infty} \frac{1}{x} = 0 = \lim_{x\to\infty} \frac{1}{x^{1/2}}$.]

[✓ **Converges**] Diverges Not possible to conclude.

(b) $\int_1^\infty h(x)dx$: [Diverges by Comparison Test, since $0 < \frac{1}{x} \leq h(x)$ and $\int_1^\infty \frac{1}{x}\,dx = \infty$.]

Converges [✓ **Diverges**] Not possible to conclude.

ii. If $p = 2$, [Since $p=2$, $0 < f(x) \leq \frac{1}{x^2}$ for $0<x\leq 1$ and $\frac{1}{x^{5/2}} \leq h(x) \leq \frac{1}{x^2}$ for $x \geq 1$.]

(a) $\int_1^\infty h(x)dx$: [Converges by Comparison Test, since $0 < h(x) \leq \frac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2}dx$ converges.]

[✓ **Converges**] Diverges Not possible to conclude.

(b) $\int_0^1 f(x)dx$

Converges Diverges [✓ **Not possible to conclude.**]

Converges if $f(x)$ is very close to 0;
diverges if $f(x) = \frac{1}{x^2}$ [redacted].
Explicit examples: converges if $f(x) = x$ [redacted] (or alternately $f(x) = x^2$).

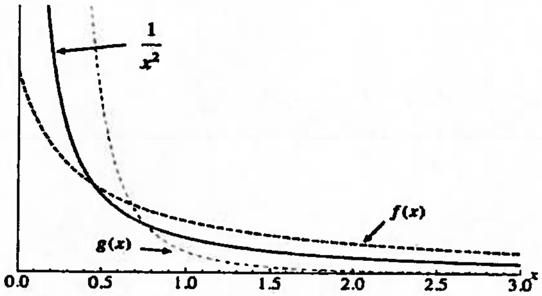
4. [13 points]

a. [8 points] Consider the functions $f(x)$ and $g(x)$ where

$$\frac{1}{x^2} \le g(x) \quad \text{for} \quad 0 < x < \frac{1}{2}.$$

$$g(x) \le \frac{1}{x^2} \quad \text{for} \quad 1 < x$$

$$\frac{1}{x^2} \le f(x) \quad \text{for} \quad 1 < x.$$



Using the information about $f(x)$ and $g(x)$ provided above, determine which of the following integrals is convergent or divergent. Circle your answers. If there is not enough information given to determine the convergence or divergence of the integral circle NI.

All we know about $f(x)$ for $1<x$ is that $\frac{1}{x^2} \le f(x)$. If $f(x) = \frac{1}{x^2}$ then $\int_1^\infty f(x)\,dx$ conv.; if $f(x)=1$ then $\int_1^\infty f(x)\,dx$ div. →

i) $\int_1^\infty f(x)dx$ CONVERGENT DIVERGENT [✓ NI]

ii) $\int_1^\infty g(x)dx$ [✓ CONVERGENT] DIVERGENT NI ← by Comp. Test, since $0 < g(x) < \frac{1}{x^2}$ for $x > 1$ (by the graph) and $\int_1^\infty \frac{1}{x^2}dx$ conv.

iii) $\int_0^1 f(x)dx$ [✓ CONVERGENT] DIVERGENT NI ↖ by the graph, $f(x)$ is bounded on $[0,1]$, so this is a proper integral which therefore converges.

iv) $\int_0^1 g(x)dx$ CONVERGENT [✓ DIVERGENT] NI

Since $0 < \frac{1}{x^2} \le g(x)$ for $0 < x < \frac{1}{2}$, and $\int_0^{1/2} \frac{1}{x^2}dx = \infty$, $\therefore \int_0^{1/2} g(x)dx = \infty$ by Comp. Test, so also $\int_0^1 g(x)dx$ diverges.

~~**b.** [5 points] Does $\int_e^\infty \frac{1}{x(\ln x)^2}dx$ converge or diverge? If the integral converges, compute its value. Show all your work. Use u substitution.~~

~~*Solution:*~~

$$\int_e^\infty \frac{1}{x(\ln x)^2}dx = \lim_{b\to\infty}\int_e^b \frac{1}{x(\ln x)^2}dx$$

~~using $u = \ln x$~~ $$= \lim_{b\to\infty}\int_1^{\ln b} \frac{1}{u^2}dx = \lim_{b\to\infty} -\frac{1}{u}\Big|_1^b = \lim_{b\to\infty} 1 - \frac{1}{\ln b} = 1$$ ~~converges.~~

6. [12 points] Determine the convergence or divergence of the following improper integrals. Justify your answers. Make sure to properly cite any results of convergence or divergence of integrals that you use. If you use the comparison test, be sure to show all your work. Circle your answer.

a. [4 points] $\int_3^\infty \frac{1}{\sqrt[3]{x}+e^{2x}}\,dx.$ CONVERGES DIVERGES

Solution: Since

$$\frac{1}{\sqrt[3]{x}+e^{2x}} \le \frac{1}{e^{2x}} = e^{-2x}$$

and $\int_3^\infty e^{-2x}dx$ converges then $\int_3^\infty \frac{1}{\sqrt[3]{x}+e^{2x}}dx$ converges.

b. [4 points] $\int_2^\infty \frac{3+b\sin^2(x^4)}{x^5}dx$, where b is a positive constant.

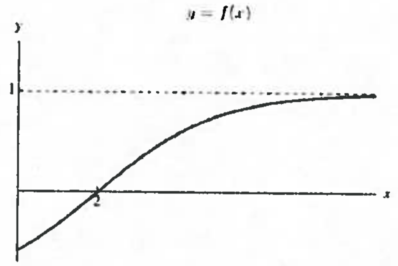
CONVERGES DIVERGES

Solution: Since

$$\frac{3+b\sin^2(x^4)}{x^5} \le (3+b)\left(\frac{1}{x^5}\right)$$

and $\int_2^\infty \frac{1}{x^5}dx$ converges ($p>1$) then $\int_2^\infty \frac{3+b\sin^2(x^4)}{x^5}dx$ converges.

c. [4 points] Let $f(x)$ be the differentiable function shown below. Note that $f(x)$ has a horizontal asymptote at $y=1$.



Does $\int_2^\infty \frac{f'(x)}{1+f(x)}dx$ converge or diverge? Circle your answer. If it converges, find its value.

CONVERGES DIVERGES

Solution:

$$\int_2^\infty \frac{f'(x)}{1+f(x)}dx = \lim_{b\to\infty}\int_2^b \frac{f'(x)}{1+f(x)}dx$$

$$= \lim_{b\to\infty} \ln|1+f(x)|\Big|_2^b = \lim_{b\to\infty}\Big(\ln|1+f(b)| - \ln|1+f(2)|\Big) = \ln 2.$$

(Note that $\lim_{b\to\infty} f(b) = 1$ and $f(2)=0$, so this is $\ln|1+1| - \ln|1+0| = \ln 2 - \ln 1 = \ln 2$.)

10. [12 points] Suppose that $g(x)$ and $h(x)$ are positive continuous functions on the interval $(0, \infty)$ with the following properties:

- $\int_1^\infty g(x)\,dx$ converges.
- $\int_0^1 g(x)\,dx$ diverges.
- $e^{-x} \le h(x) \le \frac{1}{x}$ for all x in $(0, \infty)$.

For each of the following questions, circle the correct answer.

a. [2 points] Does the integral $\int_1^\infty h(x)^2\,dx$ converge?

[✓ Converge] Diverge Cannot determine

[By Comp. Test, since $0 < h(x)^2 \le \frac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2}dx$ converges.]

b. [2 points] Does the integral $\int_0^1 h(x)\,dx$ converge?

[Converges if $h(x) = e^{-x}$ ($\int_0^1 e^{-x}dx = -e^{-x}\Big]_0^1 = 1 - \frac{1}{e}$).
Diverges if $h(x) = \frac{1}{x}$.]

Converge Diverge [✓ Cannot determine]

c. [2 points] Does the integral $\int_1^\infty h(1/x)\,dx$ converge?

[redacted]

Diverges because $\lim_{x\to\infty} h(\frac{1}{x}) = \lim_{u\to0^+} h(u) \ne 0$, since $h(x) \ge e^{-x}$ and $\lim_{x\to0^+} e^{-x} = 1 > 0$.

Converge [✓ Diverge] Cannot determine

d. [2 points] Does the integral $\int_0^1 g(x)h(x)\,dx$ converge?

[Diverges by Comp. Test, since $\frac{1}{e} \le h(x)$ for $0<x\le1$ so that [redacted] $0 < \frac{1}{e}\cdot g(x) \le g(x)\cdot h(x)$ for $0<x\le1$, and $\int_0^1 \frac{1}{e}\cdot g(x)dx$ diverges.]

Converge [✓ Diverge] Cannot determine

e. [2 points] Does the integral $\int_1^\infty g(x)h(x)\,dx$ converge?

[Converges by Comp. Test, since $0 < h(x) \le 1$ for $x \ge 1$ so that $0 < g(x)h(x) \le g(x)$ for $x \ge 1$, and $\int_1^\infty g(x)\,dx$ converges.]

[✓ Converge] Diverge Cannot determine

f. [2 points] Does the integral $\int_1^\infty e^x g(e^x)\,dx$ converge?

[✓ Converge] Diverge Cannot determine

$$\int_1^\infty e^x g(e^x)dx = \lim_{b\to\infty} \int_1^b e^x g(e^x)dx = \lim_{b\to\infty} \int_e^{e^b} g(u)du$$

$$\left[\begin{aligned} u &= e^x \\ du &= e^x dx \end{aligned}\right]$$ [redacted]

$$= \lim_{c\to\infty} \int_e^c g(u)du$$

$= \int_e^\infty g(x)dx$ converges because $1 \le e < \infty$ and $\int_1^\infty g(x)dx$ converges.

7. [8 points] Suppose that $f(x)$ is a differentiable function, defined for $x > 0$, which satisfies the inequalities $0 \le f(x) \le \frac{1}{x}$ for $x > 0$. Determine whether the following statements are always, sometimes or never true by circling the appropriate answer. No justification is necessary.

a. [2 points] $\int_1^\infty f(x)dx$ converges.

[Conv. if $f(x) = 0$.
Div. if $f(x) = \frac{1}{x}$.]

Always [✓ **Sometimes**] **Never**

b. [2 points] $\int_1^\infty (f(x))^2 dx$ converges.

[Conv. by Comp. Test, since
$0 \le f(x)^2 \le \frac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2} dx$ Conv.]

[✓ **Always**] **Sometimes** **Never**

c. [2 points] $\int_0^1 f(x)dx$ converges.

[Conv. if $f(x) = 0$.
Div. if $f(x) = \frac{1}{x}$.]

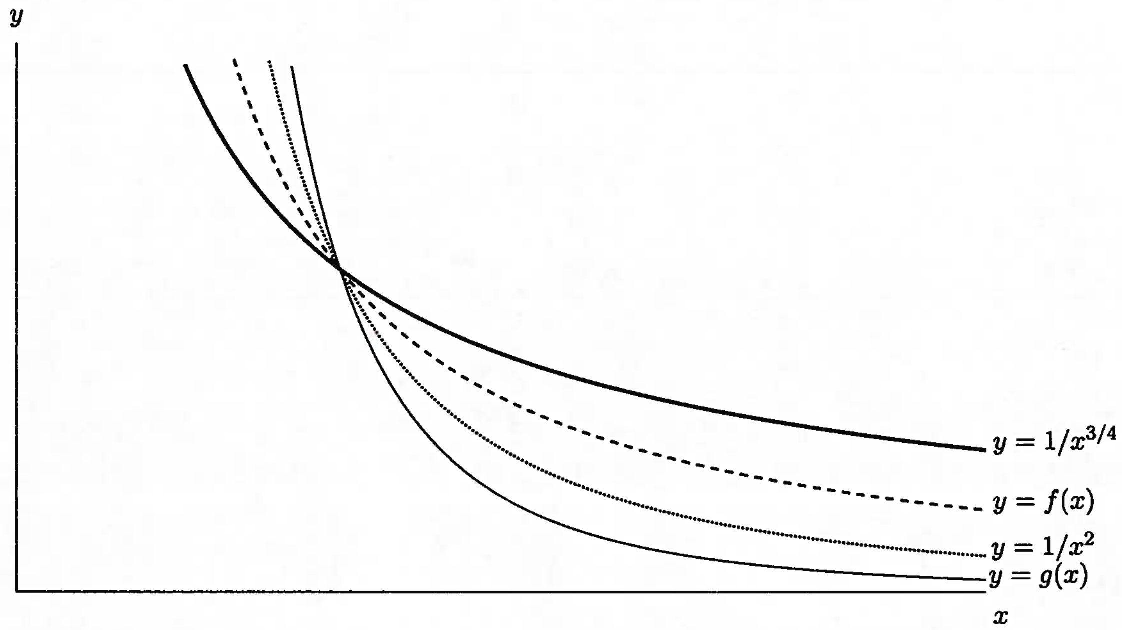
Always [✓ **Sometimes**] **Never**

d. [2 points] $\int_1^\infty e^{f(x)} dx$ converges.

[Since $0 \le f(x) \le \frac{1}{x}$ and $\lim_{x\to\infty} \frac{1}{x} = 0$,
we have $\lim_{x\to\infty} f(x) = 0$, so that
$\lim_{x\to\infty} e^{f(x)} = e^{\lim_{x\to\infty} f(x)} = e^0 = 1 \neq 0$,
$\therefore \int_1^\infty f(x)\,dx$ diverges.]

Always **Sometimes** [✓ **Never**]

10. [15 points] Consider the graph below depicting four functions for $x > 0$. The only point of intersection between any two of the functions is at $x = 1$. The functions $f(x)$ and $g(x)$ are both differentiable, and they each have $y = 0$ as a horizontal asymptote and $x = 0$ as a vertical asymptote.



Use the graph to determine whether the following quantities converge or diverge, and circle the appropriate answer. If there is not enough information to determine convergence or divergence, circle "not enough information". You do not need to show your work.

a. [3 points] $\int_1^\infty f(x)\,dx$

[Converges if $f(x)$ is close to $\frac{1}{x^2}$.
Diverges if $f(x)$ is close to $\frac{1}{x^{3/4}}$.]

Solution:

CONVERGES DIVERGES [✓ NOT ENOUGH INFORMATION]

b. [3 points] $\int_0^1 g(x)\,dx$

[Diverges by Comp. Test, since $g(x) \geq \frac{1}{x^2} > 0$ for $0 < x \leq 1$
and $\int_0^1 \frac{1}{x^2} dx = \infty$.]

Solution:

CONVERGES [✓ DIVERGES] NOT ENOUGH INFORMATION

c. [3 points] $\int_0^1 g'(x)\,e^{-g(x)}\,dx$

[Put $u = g(x)$, $du = g'(x)\,dx$, to get
$\lim_{a \to 0^+} \int_a^1 g'(x)e^{-g(x)}dx = \lim_{a\to 0^+} \int_{g(a)}^{g(1)} e^{-u}\,du$
$= \lim_{a \to 0^+} -e^{-u}\Big]_{g(a)}^{g(1)}$
$= -e^{-g(1)}$ since
$\lim_{a\to 0^+} g(a) = \infty$
so that
$\lim_{a \to 0^+} e^{-g(a)} = 0$.]

Solution:

[✓ CONVERGES] DIVERGES NOT ENOUGH INFORMATION

d. [3 points] $\int_1^\infty \sqrt{g(x)}\,dx$

Solution:

CONVERGES DIVERGES [✓ NOT ENOUGH INFORMATION]

[Diverges if $g(x)$ is close to $\frac{1}{x^2}$, since $\int_1^\infty \frac{1}{x} dx = \infty$.
Converges if $g(x)$ is close to 0 for big x.]

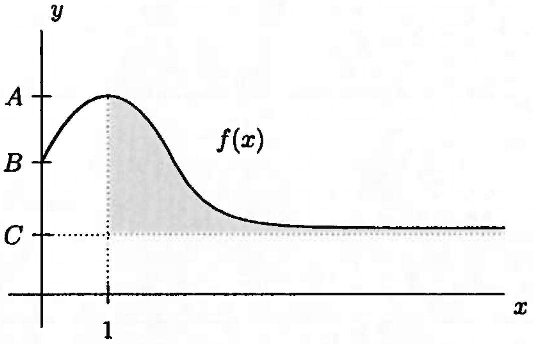
e. [3 points] The volume of the solid formed by rotating the region between $f(x)$ and the x-axis from $x = 1$ to $x = \infty$ about the x-axis

Solution:

[✓ CONVERGES] DIVERGES NOT ENOUGH INFORMATION

This volume is $\int_1^\infty \pi \cdot f(x)^2\, dx$, which converges by Comp. Test since $0 < \pi \cdot f(x)^2 \leq \left(\frac{1}{x^{3/4}}\right)^2 = \frac{1}{x^{3/2}}$ for $x \geqslant 1$, and $\int_1^\infty \frac{1}{x^{3/2}}\, dx$ converges.

10. [14 points] A function f has domain $[0,\infty)$, and its graph is given below. The numbers A, B, C are positive constants. The shaded region has **finite area**, but it extends infinitely in the positive x-direction. The line $y = C$ is a horizontal asymptote of $f(x)$ and $f(x) > C$ for all $x \geq 0$. The point $(1, A)$ is a local maximum of f.



a. [5 points] Determine the convergence of the improper integral below. **You must give full evidence supporting your answer, showing all your work and indicating any theorems about integrals you use.**

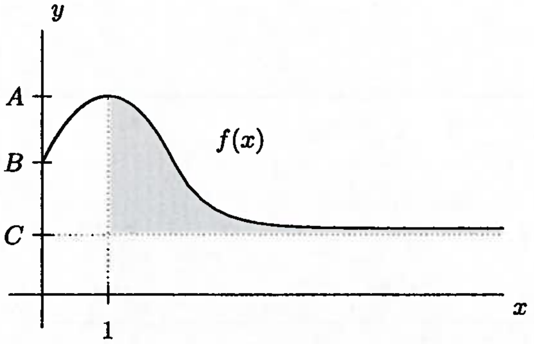
$$\int_0^1 \frac{f(x)}{x}\,dx$$

Solution: We have that for $0 < x \leq 1$

$$\frac{f(x)}{x} \geq \frac{B}{x}$$

The improper integral $\int_0^1 \frac{B}{x}\,dx = B\int_0^1 \frac{1}{x}\,dx$ diverges by the p-test with $p = 1$. Thus, the integral $\int_0^1 \frac{f(x)}{x}\,dx$ diverges by the comparison test.

10. (continued) For your convenience, the graph of f is given again. The numbers A, B, C are positive constants. The shaded region has **finite area**, but it extends infinitely in the positive x-direction. The line $y = C$ is a horizontal asymptote of $f(x)$ and $f(x) > C$ for all $x \geq 0$. The point $(1, A)$ is a local maximum of f.



$\left(\begin{array}{l} u = f(x) \\ du = f'(x)dx \end{array}\right)$

b. [3 points] **Circle** the correct answer. The value of the integral $\int_1^\infty f(x)f'(x)\,dx$

$$\lim_{b\to\infty} \int_1^b f(x)f'(x)dx = \lim_{b\to\infty} \int_{f(1)}^{f(b)} u\,du = \lim_{b\to\infty} \frac{u^2}{2}\Big]_{f(1)=A}^{f(b)} = \frac{C^2}{2} - \frac{A^2}{2}$$

is $C - A$ [✓ is $\frac{C^2 - A^2}{2}$] is $B - A$ cannot be determined diverges

c. [3 points] **Circle** the correct answer. The value of the integral $\int_1^\infty f'(x)\,dx$

$$\lim_{b\to\infty} \int_1^b f'(x)dx = \lim_{b\to\infty} f(x)\Big]_1^b = C - f(1) = C - A.$$

[✓ is $C - A$] is $\frac{C^2 - A^2}{2}$ is C cannot be determined diverges

d. [3 points] Determine, with justification, whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} (f(n) - C)$$

Solution: We notice that the function $f(x) - C$ is decreasing, (when $x \geq 1$) positive with $\lim_{x\to\infty}(f(x) - C) = 0$. By the integral test, the series

$$\sum_{n=1}^{\infty} (f(n) - C)$$

converges if and only if the improper integral

$$\int_1^\infty (f(x) - C)$$

converges. But this integral gives exactly the shaded area, which we know that it is finite. So this integral converges and therefore the series converges as well.

9. [10 points] Suppose that f is function with the following properties:

f is differentiable $\qquad f(x) > 0$ for all x $\qquad \int_1^{\infty} f(x)\,dx$ converges.

For each of the following parts, determine whether the statement is always, sometimes, or never true by circling the appropriate answer. No justification is needed.

a. [2 points] $\int_{500}^{\infty} 1000 f(x)\,dx$ converges.

[Since $1 < 500 < \infty$ and $\int_1^{\infty} f(x)\,dx$ conv., also $\int_{500}^{\infty} f(x)\,dx$ conv., so that $\int_{500}^{\infty} 1000 f(x)\,dx$ conv. as well.]

[✓ **ALWAYS**] **SOMETIMES** **NEVER**

b. [2 points] $\int_1^{\infty} (f(x))^{2/3}\,dx$ converges.

[Conv. if $f(x)$ is very close to 0, such as $f(x) = \frac{1}{x^2}$
Div. if $f(x) = \frac{1}{x^{3/2}}$.]

ALWAYS [✓ **SOMETIMES**] **NEVER**

c. [2 points] $\int_1^{\infty} (f(x))^{3/2}\,dx$ converges.

[Since $\int_1^{\infty} f(x)\,dx$ conv., $\lim\limits_{x\to\infty} f(x)$ must be 0, so that $\lim\limits_{x\to\infty} \frac{f(x)^{3/2}}{f(x)} = 0$. By LCT for improper integrals, since $f(x) > 0$ and $f(x)^{3/2} > 0$ and $\int_1^{\infty} f(x)\,dx$ conv. $\therefore \int_1^{\infty} f(x)^{3/2}\,dx$ conv.

[✓ **ALWAYS**] **SOMETIMES** **NEVER**

d. [2 points] $\int_0^1 f\left(\frac{1}{x}\right)\,dx$ converges.

$\left(\begin{array}{l} u = \frac{1}{x} \\ du = -\frac{1}{x^2}dx = -u^2 dx \end{array}\right)$

[$\lim\limits_{a\to 0^+} \int_a^1 f\left(\frac{1}{x}\right)dx = \lim\limits_{a\to 0^+} \int_{\frac{1}{a}}^{1} \frac{f(u)}{-u^2}\,du$

$= \lim\limits_{a\to 0^+} \int_1^{1/a} \frac{f(u)}{u^2}\,du = \lim\limits_{b\to\infty} \int_1^{b} \frac{f(u)}{u^2}\,du$

"$\int_1^{\infty} \frac{f(u)}{u^2}\,du$" which conv. by Comp. Test since $0 < \frac{f(u)}{u^2} \le f(u)$ if $u \geq 1$, and $\int_1^{\infty} f(x)\,dx$ conv.]

[✓ **ALWAYS**] **SOMETIMES** **NEVER**

e. [2 points] $\int_1^{\infty} \frac{f'(x)}{f(x)}\,dx$ converges. $\qquad \left(\text{Note: } \frac{f'(x)}{f(x)} = \frac{d}{dx}\ln(f(x)). \right)$

ALWAYS **SOMETIMES** [✓ **NEVER**]

$\lim\limits_{b\to\infty} \int_1^b \frac{f'(x)}{f(x)}\,dx = \lim\limits_{b\to\infty} \ln(f(x))\Big]_1^b = -\infty$

since convergence of $\int_1^{\infty} f(x)\,dx$ implies that $\lim\limits_{x\to\infty} f(x) = 0$

so that $\lim\limits_{b\to\infty} \ln(f(b)) = \lim\limits_{c\to 0^+} \ln(c) = -\infty$.

5. [10 points] Let $f(x)$ and $g(x)$ be two functions that are differentiable on $(0,\infty)$ with continuous derivatives and which satisfy the following inequalities for all $x \geq 1$:

$$\frac{1}{x} \leq f(x) \leq \frac{1}{x^{1/2}} \quad \text{and} \quad \frac{1}{x^2} \leq g(x) \leq \frac{1}{x^{3/4}}.$$

For each of the following, determine whether the integral always, sometimes, or never converges. Indicate your answer by circling the one word that correctly fills the answer blank. No justification is necessary. No credit will be awarded for unclear markings.

a. [2 points] $\int_1^\infty \sqrt{f(x)}\,dx$ __________ converges.

[Diverges by Comp. Test, since $0 < \frac{1}{x^{1/2}} \leq \sqrt{f(x)}$ and $\int_1^\infty \frac{1}{x^{1/2}}dx = \infty$.]

Always Sometimes [✓ Never]

b. [2 points] $\int_3^\infty 4000g(x)\,dx$ __________ converges.

[Conv. if $g(x) = \frac{1}{x^2}$; div. if $g(x) = \frac{1}{x^{3/4}}$.]

Always [✓ Sometimes] Never

c. [2 points] $\int_1^\infty f(x)g(x)\,dx$ __________ converges.

[redacted]

[✓ Always] Sometimes Never

[Conv. by Comp. Test, since $0 < f(x)g(x) \leq x^{\frac{1}{2}+\frac{3}{4}}$ and $\int_1^\infty \frac{1}{x^{\frac{1}{2}+\frac{3}{4}}}dx$ conv. since $\frac{1}{2}+\frac{3}{4} > 1$.]

d. [2 points] $\int_5^\infty g'(x)e^{g(x)}\,dx$ __________ converges.

[✓ Always] Sometimes Never

[This is [redacted] $\lim_{b\to\infty}$ [redacted] $\int_5^b g'(x)e^{g(x)}dx$

$= \lim_{b\to\infty} e^{g(x)}\Big]_5^b$

$= e^{\lim_{b\to\infty} g(b)} - e^{g(5)} = e^0 - e^{g(5)}$]

I just guessed an antiderivative. You could [redacted] substitute $u=g(x)$ to find this.

e. [2 points] $\int_1^\infty f'(x)\ln(f(x))\,dx$ __________ converges.

[✓ Always] Sometimes Never

$\lim_{b\to\infty} \int_1^b f'(x)\ln(f(x))dx = \lim_{b\to\infty} \int_{f(1)}^{f(b)} \ln u\, du = \lim_{b\to\infty} (u\ln u - u)\Big]_{f(1)=1}^{f(b)}$

(integrate by parts)

[$u = f(x)$
$du = f'(x)dx$]

$= $ (since $\lim_{x\to\infty} f(x) = 0$) $\lim_{c\to 0^+} (c\ln c) + 1 = 1$

(by l'Hôpital)