Math 116 — Practice for Exam 3

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Name: SOLUTIONS

Instructor: ______________________ Section Number: ________

1. This exam has 4 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2012	3	2		13	
Winter 2013	3	2		6	
Fall 2013	3	4		9	
Fall 2016	3	3		8	
Total				36	

Recommended time (based on points): 43 minutes

2. [13 points] Determine if each of the following sequences is increasing, decreasing or neither, and whether it converges or diverges. Circle all the answers that apply. On parts **a-c**, if the sequence converges, find the limit. No justification is required.

a. [3 points] For $n \geq 1$, let $a_n = 3 + \frac{1}{n}$.

Solution:

1. Increasing **Decreasing** Neither.
2. **Convergent**: $\lim_{n\to\infty} a_n = 3$ Divergent

b. [3 points] For $n \geq 1$, let $a_n = (-\frac{\pi}{e})^n$.

Solution:

1. Increasing Decreasing **Neither.**
2. Convergent: $\lim_{n\to\infty} a_n =$ __________ **Divergent**

c. [3 points] Let $P(x)$ be the cumulative distribution function of a nonzero probability density function $p(x)$. Define $a_n = P(n)$ for $n \geq 1$.

Solution:

1. **Increasing** Decreasing Neither.
2. **Convergent**: $\lim_{n\to\infty} a_n = 1$ Divergent

d. [2 points] For $n \geq 1$, let $a_n = 1 - \dfrac{1}{1!} + \dfrac{1}{2!} - \dfrac{1}{3!} + \cdots + \dfrac{(-1)^n}{n!}$.

Solution:

1. Increasing Decreasing **Neither**.
2. **Convergent** (no need to compute the limit) Divergent

e. [2 points] Let $a_n = \displaystyle\int_2^n \frac{1}{\sqrt{x}-1}dx$, for $n \geq 2$.

Solution:

1. **Increasing** Decreasing Neither.
2. Convergent (no need to compute the limit) **Divergent**

2. [6 points] Let the sequence a_n be given by

$$a_1 = -1, \quad a_2 = \frac{\sqrt{2}}{3}, \quad a_3 = -\frac{\sqrt{3}}{5}, \quad a_4 = \frac{\sqrt{4}}{7}, \quad a_5 = -\frac{\sqrt{5}}{9}, \quad a_6 = \frac{\sqrt{6}}{11}$$

a. [1 point] Find a_7.

Solution:

$$a_7 = \frac{-\sqrt{7}}{13}.$$

b. [3 points] Write a formula for a_n.

Solution:

$$a_n = (-1)^n \frac{\sqrt{n}}{2n-1}.$$

c. [2 points] Does the sequence a_n converge? If so, find its limit.

Solution: Yes, it converges to 0.

4. [9 points]
Determine if each of the following sequences is increasing, decreasing or neither, and whether it converges or diverges. If the sequence converges, identify the limit. Circle all your answers. No justification is required.

a. [3 points] $a_n = \displaystyle\int_1^{n^3} \frac{1}{(x^2+1)^{\frac{1}{5}}}\,dx.$

Converges to ______________ **DIVERGES**.

INCREASING Decreasing Neither.

Solution: (Not required)
Since $\displaystyle\int_1^{\infty} \frac{1}{(x^2+1)^{\frac{1}{5}}}\,dx \approx \int_1^{\infty} \frac{1}{x^{\frac{2}{5}}}\,dx$ which diverges by p-test (with $p = \frac{2}{5} \le 1$). Hence, $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \int_1^{n^3} \frac{1}{(x^2+1)^{\frac{1}{5}}}\,dx = \int_1^{\infty} \frac{1}{(x^2+1)^{\frac{1}{5}}}\,dx = \infty$. The sequence is increasing since $\displaystyle\frac{1}{(x^2+1)^{\frac{1}{5}}} > 0$.

b. [3 points] $b_n = \displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}.$

CONVERGES TO $\sin 1$ Diverges.

Increasing Decreasing **NEITHER**.

Solution: (Not required)
Since $\sin x = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, then $\displaystyle\lim_{n\to\infty} b_n = \sum_{k=0}^{\infty} \frac{(-1)^k (1)^{2k+1}}{(2k+1)!} = \sin 1$. Since the series is alternating, the sequence is neither increasing or decreasing.

c. [3 points] $c_n = \cos(a^n)$, where $0 < a < 1$.

CONVERGES TO 1 Diverges.

INCREASING Decreasing Neither.

Solution: (Not required)
Since $\displaystyle\lim_{n\to\infty} a^n = 0$, then $\displaystyle\lim_{n\to\infty} \cos a^n = \cos 0 = 1$.

3. [8 points] For $n = 1, 2, 3, \ldots$ consider the sequence a_n given by

$$a_n = \frac{-1}{2^{(n+1)/2}} \text{ if } n \text{ is odd}, \qquad a_n = \frac{1}{3^{n/2}} \text{ if } n \text{ is even}.$$

a. [2 points] Write out the first 5 terms of the sequence a_n.

Solution: The first five terms are

$$-\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{8}.$$

b. [2 points] The series $\sum_{n=1}^{\infty} a_n$ is alternating. In a sentence or two, explain why the Alternating Series Test **cannot** be used to determine whether $\sum_{n=1}^{\infty} a_n$ converges or diverges.

Solution: The condition $|a_{n+1}| < |a_n|$ does not hold for all n. (It does not even hold eventually.)

c. [4 points] The series $\sum_{n=1}^{\infty} a_n$ converges. Show that it converges, either by using theorems about series, or by computing its exact value.

Solution: One possible answer is that the series is equal to the difference of two convergent geometric series:

$$\sum_{k=1}^{\infty} \frac{1}{3^k} - \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{\frac{1}{3}}{1-\frac{1}{3}} - \frac{\frac{1}{2}}{1-\frac{1}{2}} = -\frac{1}{2}.$$

Another answer uses the Comparison Test; for $n = 1, 2, \ldots$, let $b_n = \frac{1}{n^2}$, and notice that $|a_n| \leq b_n$ eventually. Since $\sum_{n=1}^{\infty} b_n$ converges by the p-Test ($p = 2$), $\sum_{n=1}^{\infty} |a_n|$ converges by comparison. Hence the original series converges.