Math 116 — Practice for Exam 3

Generated November 29, 2017

Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 4 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2015	3	3	rumor	13	
Winter 2015	2	6	caffeine drip	9	
Winter 2012	2	5		13	
Fall 2015	2	5		10	
Total				45	

Recommended time (based on points): 45 minutes

3. [13 points]

a. [4 points] The number of people R that have heard a rumor increases at a rate proportional to the product of the number of people that have heard the rumor and the number of people that haven't yet heard the rumor. Write a differential equation for R which models the scenario described assuming that the total number of people is 1,000. Use $k > 0$ for the constant of proportionality.

Solution: The number of people that have heard the rumor is R, so the number of people that have not yet heard the rumor is $1000 - R$. Thus the differential equation is $\frac{dR}{dt} = kR(1000 - R)$.

$$\frac{dR}{dt} = \underline{\quad kR(1000 - R) \quad}$$

b. [4 points] For what values of A, B is $y(t) = At\cos t + Bt$ a solution to the differential equation $ty' = y + t^2 \sin t$ satisfying the initial condition $y\left(\frac{\pi}{2}\right) = 2\pi$? Be sure to show your work.

Solution: Since $y'(t) = A\cos t - At\sin t + B$, $y(t)$ is a solution if

$$t(A\cos t - At\sin t + B) = At\cos t + Bt + t^2 \sin t \qquad \Rightarrow \qquad -At^2 \sin t = t^2 \sin t$$

Thus $A = -1$. Plugging in the initial condition $y\left(\frac{\pi}{2}\right) = 2\pi$,

$$-\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + B\left(\frac{\pi}{2}\right) = 2\pi.$$

Thus $B = 4$.

$A =$ **-1**

$B =$ **4**

c. [5 points] Find the solution to the differential equation

$$e^{-x} + y^2 \frac{dy}{dx} = 0, \quad \text{with initial condition} \quad y(0) = 2.$$

Solution: Moving the e^{-x} to the right side of the equation and separating variables,

$$\int y^2 dy = \int -e^{-x} dx$$

$$\frac{1}{3}y^3 = e^{-x} + C$$

$$y = \sqrt[3]{3e^{-x} + C}$$

Plugging in the initial condition $y(0) = 2$, $2 = \sqrt[3]{3 + C}$. Therefore $C = 5$.

$$y = \underline{\quad \sqrt[3]{3e^{-x} + 5} \quad}$$

6. [9 points] An extremely sleepy graduate student is grading Math 116 exams. She has been drinking coffee all day, but it just is not enough. She hooks up a caffeine drip that delivers caffeine into her body at a constant rate of 170 mg/hr. The amount of caffeine in her body decays at a rate proportional to the current amount of caffeine in her body. The half-life of caffeine in her body is 6 hours.

a. [4 points] Using the blank provided, write a differential equation which models the scenario described above. Use $Q(t)$ for the amount of caffeine in the graduate student's body, measured in mg, t for hours after she hooked up the caffeine drip, and $k > 0$ for the constant of proportionality.

Solution: The rate that the amount of caffeine in the graduate student's body is changing over time should be the rate that caffeine is entering their body minus the rate that caffeine is leaving their body. The rate that caffeine is entering the graduate students body is a constant 170 mg/hr. The rate that caffeine is leaving the graders body is proportional to the current amount, so it is kQ mg/hr. Putting all this together gives us the equation written below.

$$\frac{dQ}{dt} = \underline{\qquad 170 - kQ \qquad}$$

b. [5 points] Use the half-life of caffeine to determine the constant of proportionality.

Solution: We know that the amount of caffeine in the graduate student's body decays exponentially with decay rate k. If C_0 is the initial amount of caffeine, then a half-life of 6 hours means that

$$\frac{1}{2}C_0 = C_0 e^{-k6}.$$

Solving for k gives us that $k = -\frac{1}{6}\ln\left(\frac{1}{2}\right)$.

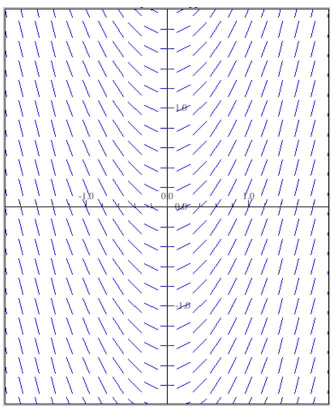
5. [13 points] Consider the following differential equations

A. $y' = 2x$ B. $y' = 5y - 1$ C. $yy' = 2$ D. $y' = \frac{y}{x}$

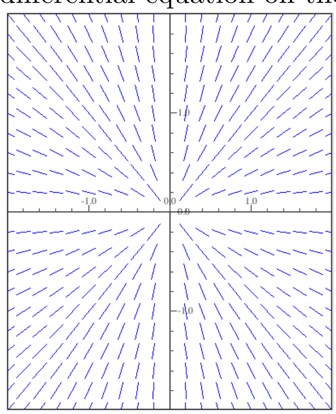
a. [6 points] Each of the following functions is a solution to one of the differential equations listed above. Indicate which differential equation with the corresponding letter (A,B,C or D) on the given line.

1. $y = \frac{1}{5} + e^{5x}$ ______B_____
2. $y = x^2 + 1$ ______A_____
3. $y = 2\sqrt{x}$ ______C_____

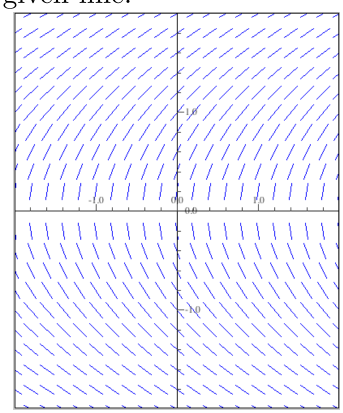
b. [3 points] Each of the following slope fields belongs to one of the differential equations listed above. Indicate which differential equation on the given line.



______A_____



______D_____

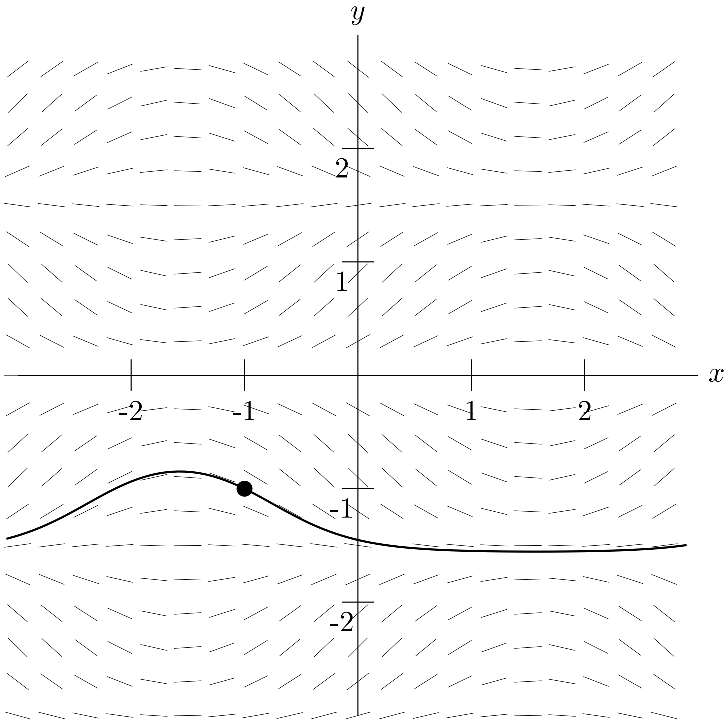


______C_____

c. [4 points] Find the equilibrium solutions of the differential equations given above (if any). Write the equation of the equilibrium solutions in the space provided below. If the equation does not have equilibrium solutions, write none.

A. ___None___ B. ___$y = \frac{1}{5}$___ C. ___None___ D. ___$y = 0$___

5. [10 points] The graph of a slope field corresponding to a differential equation is shown below.



a. [4 points] On the slope field, carefully sketch a solution curve passing through the point (-1,-1).

Solution: See graph above.

b. [2 points] The slope field pictured above is the slope field for one of the following differential equations. Which one? Circle your answer. You do not need to show your work.

$$\frac{dy}{dx} = \cos x \cos(2y) \qquad \frac{dy}{dx} = \sin x \cos(2y)$$

$$\boxed{\frac{dy}{dx} = \cos x \sin(2y)} \qquad \frac{dy}{dx} = \sin x \sin(2y)$$

c. [4 points] Find two equilibrium solutions to the differential equation you circled.

Solution: The equilibrium solutions of $\frac{dy}{dx} = \cos x \sin(2y)$ are the values of y such that $\sin(2y) = 0$. Solving, we see that the equilibrium solutions are $y = 0, \pm\frac{\pi}{2}, \pm\pi, \ldots$