Math 116 — Practice for Exam 1

Generated September 21, 2017

Name: SOLUTIONS

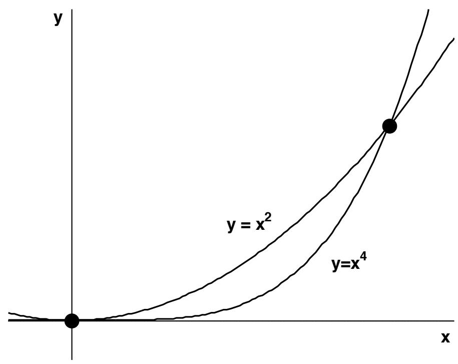
Instructor: ______________________ Section Number: __________

1. This exam has 8 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2010	1	10		15	
Winter 2010	3	4		8	
Winter 2013	1	7		8	
Fall 2014	1	7	board game	13	
Winter 2015	3	8		10	
Winter 2016	1	7	jewelry	10	
Winter 2017	1	6	Rodin Coil	10	
Fall 2003	3	3	snowman	10	
Total				84	

Recommended time (based on points): 84 minutes

10. [15 points] Consider the area between the curves $y = x^2$ and $y = x^4$ in the positive quadrant as shown in the graph below. Use this area to answer the following questions.



a. [5 points] Set up, but do not evaluate, a definite integral that describes the area described above. Write your final answer on the space provided.

$\int_0^1 (x^2 - x^4)dx$ **or** $\int_0^1 (y^{1/4} - y^{1/2})dy$

b. [5 points] Set up, but do not evaluate, a definite integral that describes the volume of the solid generated by revolving the area described above about the line $y = 2$. Write your final answer on the space provided.

$\int_0^1 \pi((2 - x^4)^2 - (2 - x^2)^2)dx$

c. [5 points] Set up, but do not evaluate, a definite integral that describes the volume of the solid whose base is the area described above and whose cross-sections perpendicular to the x-axis are squares.

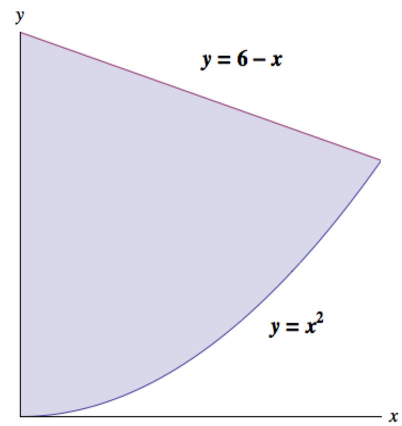
$\int_0^1 (x^2 - x^4)^2 dx$

4. [8 points] Consider a solid whose base is contained between the curves $y = e^x$, $y = 1$, and $x = 3$. Cross-sectional slices perpendicular to the x-axis are rectangles, having length contained in the base region mentioned above and height determined by $g(x) = x^2$. Determine the exact volume of this solid.

Solution: The slice has volume $x^2(e^x - 1)\Delta x$. Summing the slices and letting Δx go to 0, we have

$$\begin{aligned}
\text{Volume} &= \int_0^3 x^2(e^x - 1)dx \\
&= \int_0^3 x^2 e^x dx - \int_0^3 x^2 dx \\
&= (x^2e^x|_0^3 - \int_0^3 2xe^x dx) - \frac{1}{3}x^3|_0^3 \\
&= (x^2e^x|_0^3 - (2xe^x|_0^3 - \int_0^3 2e^x dx)) - \frac{1}{3}x^3|_0^3 \\
&= (x^2e^x - 2xe^x + 2e^x - \frac{1}{3}x^3)|_0^3 \\
&= 9e^3 - 6e^3 + 2e^3 - 9 - 2 \\
&= 5e^3 - 11
\end{aligned}$$

7. [8 points] Let S be the solid whose base is the region bounded by the curves $y = x^2$, $y = 6 - x$ and $x = 0$ and whose cross sections **parallel** to the x-axis are squares. Find a formula involving definite integrals that computes the volume of S.



Solution: First we solve for where the two curves intersect. $6-x = x^2$ implies $0 = x^2+x-6 = (x+3)(x-2)$, so $x = 2$, which implies $y = 4$. We have to split the problem into two cases, one when $0 \le y \le 4$, and one when $4 \le y \le 6$. We will choose thin horizontal slices, and integrate in terms of y, so we need to solve for x in terms of y: $x = \sqrt{y}$ and $x = 6 - y$ are our two curves.

In the case of $0 \le y \le 4$, a thin slice has volume $V_{slice} \approx (\sqrt{y})^2 \Delta y$. In the second case, $4 \le y \le 6$, a thin slice has volume $V_{slice} \approx (6-y)^2 \Delta y$. Hence the total volume of the solid is given by

$$V = \int_0^4 y dy + \int_4^6 (6-y)^2 dy.$$

7. [13 points] Kazilla is designing a new board game. She is interested in using the region R in the xy-plane bounded by $y = 2$, $y = x$, $x = 1$ and $x = 0$.

a. [4 points] The first part of the game is a spinning top formed by rotating the region R around the y-axis. Write an integral (or a sum of integrals) that gives the volume of the spinning top. Do not evaluate your integral(s).

Solution: Shell method:

$$\int_0^1 2\pi(2-x)x dx$$

Washer method:

$$\int_0^1 \pi y^2 dy + \int_1^2 \pi dy$$

b. [4 points] Another game piece has a base in the shape R, but with semicircular cross sections **perpendicular** to the x-axis. Write an integral which gives the volume of the game piece. Do not evaluate your integral.

Solution:

$$\frac{\pi}{8}\int_0^1 (2-x)^2 dx$$

c. [5 points] A third game piece has volume given by $\int_0^2 \pi(h(x))^2 dx$ where $h(x)$ is a continuous function of x. Use MID(3) to approximate the volume of this third game piece. Be sure to write out all of the terms in your approximation. Your answer may contain the function $h(x)$.

Solution:

$$MID(3) = \frac{2}{3}[\pi(h(1/3))^2 + \pi(h(1))^2 + \pi(h(5/3))^2]$$

8. [10 points] Consider the region A in the xy-plane bounded by $y = 1 - x^4$, the y-axis, and the x-axis in the first quadrant. The area of A is $\frac{4}{5}$.

a. [5 points] Suppose N is any positive whole number. Put the following quantities in order from least to greatest. MID(N), TRAP(N), RIGHT(N), LEFT(N), and the number $\frac{4}{5}$, where all of the approximations listed are for the integral $\int_0^1 (1 - x^4)\, dx$.

Solution:

RIGHT(N) $\leq$ **TRAP(N)** $\leq$ 4/5 $\leq$ **MID(N)** $\leq$ **LEFT(N)**

b. [5 points] Write an expression involving integrals that gives the volume of the solid formed by rotating the region A around the y-axis.

Solution: The volume of the solid formed by rotating the region A about the y-axis is $\int_0^1 2\pi x(1 - x^4) dx$ or equivalently $\int_0^1 \pi(1 - y)^{\frac{1}{2}} dy$

7. [10 points] Maize and Blue Jewelry Company is trying to decide on a design for their signature aMaize-ing bracelet. There are two possible designs: type W and type J. The company has done research and the two bracelet designs are equally pleasing to customers. The design for both rings starts with the function $C(x) = \cos\left(\frac{\pi}{2}x\right)$ where all units are in millimeters. Let R be the region enclosed by the graph of $C(x)$ and the graph of $-C(x)$ for $-1 \leq x \leq 1$.

a. [5 points] The type W bracelet is in the shape of the solid formed by rotating R around the line $x = 50$. Write an integral that gives the volume of the type W bracelet. Include **units**.

Solution: The volume of the type W bracelet, in mm^3, using the shell method, is

$$\int_{-1}^{1} 2\pi(50 - x) \cdot 2C(x)\, dx.$$

b. [5 points] The type J bracelet is in the shape of the solid formed by rotating R around the line $y = -50$. Write an integral that gives the volume of the type J bracelet. Include **units**.

Solution: The volume of the type J bracelet, in mm^3, using the washer method, is

$$\int_{-1}^{1} \pi(50 + C(x))^2 - \pi(50 - C(x))^2\, dx.$$

6. [10 points] The Rodin Coil is a fantastic device that (supposedly) creates unlimited free energy. The rate at which it creates this energy is a function of the volume of the coil.

a. [5 points] Suppose a prototype of the Rodin Coil is the solid whose base is the circle $x^2+y^2 = 2$ (where x and y are measured in meters), and whose cross sections perpendicular to the x–axis are squares. Write, but do **not** compute, an expression involving one or more integrals which gives the volume, in cubic meters, of this prototype.

Solution: We find that the sidelength of the square at x–coordinate x is $2 \cdot \sqrt{2-x^2}$. So the volume of a slice of thickness Δx at that point is approximately $\left(2 \cdot \sqrt{2-x^2}\right)^2 \cdot \Delta x$. Integrating from the left end of the circle to the right end we find a total volume (in cubic meters) of

$$\int_{-\sqrt{2}}^{\sqrt{2}} \left(2\sqrt{2-x^2}\right)^2 \, dx.$$

b. [5 points] One of Rodin's students was able to come up with an even more efficient free energy machine. Suppose the student's prototype was made by taking the same circle $x^2 + y^2 = 2$ and rotating it around the vertical line $x = 3$. Write, but do **not** compute, an expression involving one or more integrals which gives the volume, in cubic meters, of this prototype.

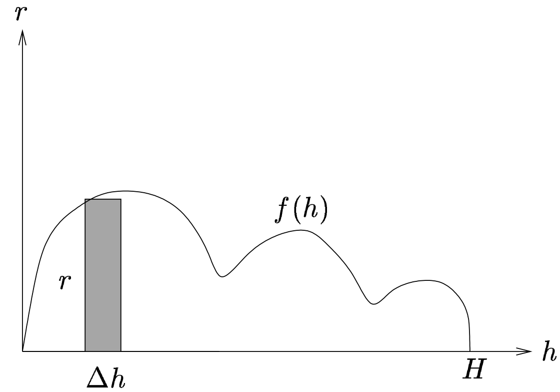
Solution:
One Solution: Using cylindrical shells perpendicular to the x–axis, we find that the volume is equal to

$$\int_{-\sqrt{2}}^{\sqrt{2}} 2\pi(3-x)\left(2\sqrt{2-x^2}\right) \, dx.$$

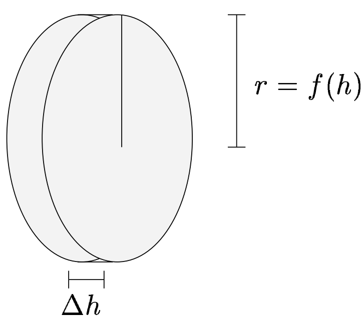
Another Solution: Using slices of the solid perpendicular to the y–axis ("washers"), we find that the volume is equal to

$$\int_{-\sqrt{2}}^{\sqrt{2}} \pi \left[\left(3+\sqrt{2-y^2}\right)^2 - \left(3-\sqrt{2-y^2}\right)^2\right] \, dy.$$

3. (10 points) The cross-sections of a snowman are given by circles of radius $r = f(h)$ where h is the height measured from the ground and $f(h)$ has graph given in the figure. Both r and h are measured in inches.



(a) Draw and label a typical, thin, cross-section of the snowman. What is the volume of the cross-section (in terms of the function $f(h)$)?



The volume of a typical cross-section can now be read directly from the picture to be

$$V_{\text{slice}} = \pi r^2 \Delta h = \pi [f(h)]^2 \Delta h.$$

(b) Write an integral in terms of $f(h)$ whose value is the total amount of snow used in making the snowman.
To find the total amount of snow used to make the snowman, we add up the volumes of all of the slices forming a Riemann sum, and then let $\Delta h \to 0$. The resulting integral is

$$\int_0^H \pi [f(h)]^2 \, dh.$$