On my honor, as a student,
I have neither given nor received
unauthorized aid on this academic work. **Initials:** ______

Do not write in this area

SOLUTIONS

Math 116 — ~~First~~ Practice for Second Midterm — ~~October 9, 2017~~

Your Initials Only: MZ Your U-M ID # (not uniqname): ??

Instructor Name: Zieve Section #: ??

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	9	
3	8	
4	12	
5	10	
Total	50	

Problem	Points	Score
6	8	
7	12	
8	10	
9	8	
10	12	
Total	100	

1. Determine the interval of convergence of each series, with full justification.

[5 points] (a) $\sum_{n=0}^{\infty} \frac{(n+2)x^n}{n^4+1}$ [you may assume that the radius of convergence is 1]

When $x=1$ or $x=-1$, $\sum_{n=0}^{\infty} \left| \frac{(n+2)x^n}{n^4+1} \right| = \sum_{n=0}^{\infty} \frac{n+2}{n^4+1}$ converges

by LCT with the convergent p-series $\sum_{n=0}^{\infty} \frac{1}{n^3}$,

since $\lim_{n\to\infty} \frac{\frac{n+2}{n^4+1}}{n^3} = 1$.

Thus $\sum_{n=0}^{\infty} \frac{(n+2)x^n}{n^4+1}$ is absolutely convergent when $x = \pm 1$,

so it's convergent.

$\therefore$ Interval of convergence is $[-1, 1]$.

[6 points] (b) $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} x^{2n}$

Be able to do this →

Ratio Test: (if $x \neq 0$) $\lim_{n\to\infty} \left| \frac{\frac{(2(n+1))!}{((n+1)!)^2} x^{2(n+1)}}{\frac{(2n)!}{(n!)^2} x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{(2n+2)!}{(2n)!} \cdot \left(\frac{n!}{(n+1)!}\right)^2 \cdot x^2 \right|$

$$= \lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)}{(n+1)^2} \cdot x^2 \right|$$

$$= 4x^2,$$

so the series converges if $|4x^2| < 1$, i.e., $|x| < \frac{1}{2}$,

and diverges if $|4x^2| > 1$, i.e., $|x| > \frac{1}{2}$.

This was a bit tricky.

If $|x| = \frac{1}{2}$ then $\left| \frac{(2n)!}{(n!)^2} \cdot x^{2n} \right| = \frac{(2n)\cdot(2n-1)\cdot \ldots \cdot (n+1)}{n \cdot (n-1) \cdot \ldots \cdot 1} \cdot \frac{1}{2^n}$

$$= \frac{2n}{2\cdot n} \cdot \frac{2n-1}{2\cdot(n-1)} \cdot \frac{2n-2}{2\cdot(n-2)} \cdot \ldots \cdot \frac{n+1}{2\cdot 1}$$

$$> 1,$$

so the series diverges by n^{th} term test.

$\therefore$ Interval of convergence is $\boxed{\left(-\frac{1}{2}, \frac{1}{2}\right)}$.

(2) Determine, with full justification, whether each series is absolutely convergent, conditionally convergent, or divergent.

[4 points] (a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2+\cos^2 n}$ diverges absolutely converges conditionally converges

(circle one)

Diverges by n^{th} term test, since $\left|\frac{(-1)^n}{2+\cos^2 n}\right| \geq \frac{1}{3}$

so $\lim_{n\to\infty} \frac{(-1)^n}{2+\cos^2 n} \neq 0.$

[5 points] (b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ diverges absolutely converges conditionally converges

(circle one)

Converges by AST, since $a_n := \frac{1}{n \ln n}$ satisfies $0 < a_{n+1} < a_n$ for $n \geq 2$

and $\lim_{n\to\infty} a_n = 0.$

Not absolutely convergent due to Integral Test,

since $f(x) := \frac{1}{x \ln x}$ is positive and decreasing for $x \geq 2$

and $\int_2^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_2^b f(x)\,dx = \lim_{b\to\infty} \ln \ln x \Big]_2^b = \infty.$

(3) ~~12~~. [8 points] Suppose that the power series $\sum_{n=0}^{\infty} a_n(x-4)^n$ converges when $x = 0$ and diverges when $x = 9$. In this problem, you do not need to show your work.

a. [4 points] Which of the following could be the interval of convergence? Circle all that apply.

$[0,8)$ $\qquad$ $[0,7]$ $\qquad$ $(-1,9)$ $\qquad$ $(-2,10)$ $\qquad$ $(0,8]$

(Justification not needed for full credit.)

The only conditions are that it is an interval centered at 4 which contains 0 (and hence contains $[0,8)$) but does not contain 9 (and hence is contained in $[-1,9)$). But $[0,7]$ is not centered at 4, $(-2,10)$ contains 9, and $(0,8]$ does not contain 0.

b. [2 points] The limit of the sequence a_n is 0.

ALWAYS $\qquad$ SOMETIMES $\qquad$ NEVER

(Justification not needed for full credit.)

n^{th} term test, and convergence when $x=0$, implies that $\lim_{n\to\infty} a_n \cdot (0-4)^n = 0$, so (since $4^n \geq 1$) it follows that $\lim_{n\to\infty} a_n = 0$.

c. [2 points] The series $\sum_{n=0}^{\infty}(-5)^n a_n$ converges.

ALWAYS $\qquad$ SOMETIMES $\qquad$ NEVER

(Justification not needed for full credit.)

$\sum_{n=0}^{\infty} (-5)^n a_n$ is the value of the power series when $x=-1$, so it converges precisely when -1 is contained in the interval of convergence of the power series.

Examples: if $a_n = \frac{1}{(n+1)\cdot 5^n}$ then $\sum_{n=0}^{\infty} (-5)^n a_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges by AST (but $\sum_{n=0}^{\infty} a_n (x-4)^n$ diverges when $x=9$ + converges when $x=0$).

If $a_n = \frac{1}{(n+1)\cdot 4^n}$ then $\sum_{n=0}^{\infty} (-5)^n a_n$ diverges by n^{th} term test (or Ratio Test) but $\sum_{n=0}^{\infty} a_n (x-4)^n$ diverges when $x=9$ + conv. when $x=0$.

4 ~~8~~. [12 points] Suppose a_n and b_n are sequences of positive numbers with the following properties.

- $\sum_{n=1}^{\infty} a_n$ converges.
- $\sum_{n=1}^{\infty} b_n$ diverges.
- $0 < b_n \le M$ for some positive number M.

For each of the following questions, circle the correct answer. No justification is necessary.

a. [2 points] Does the series $\sum_{n=1}^{\infty} a_n b_n$ converge?

$\boxed{\text{Converge}}$ Diverge Cannot determine

By Comparison Test, since $0 < a_n b_n \le M a_n$ and $\sum_{n=1}^{\infty} M a_n$ converges.

b. [2 points] Does the series $\sum_{n=1}^{\infty} (-1)^n b_n$ converge?

Converge Diverge $\boxed{\text{Cannot determine}}$

Diverges if $b_n = 1$ for each n. Converges if $b_n = \frac{1}{n}$ for each n.

c. [2 points] Does the series $\sum_{n=1}^{\infty} \sqrt{b_n}$ converge?

Converge $\boxed{\text{Diverge}}$ Cannot determine

If it converged then n^{th} term test implies $\lim_{n\to\infty} \sqrt{b_n} = 0$, so $0 < b_n < 0.001$ for big n, whence $0 < b_n < \sqrt{b_n}$ for big n so $\sum_{n=1}^{\infty} b_n$ would converge by Comparison Test. But $\sum_{n=1}^{\infty} b_n$ diverges by hypothesis, so $\sum_{n=1}^{\infty} \sqrt{b_n}$ diverges as well.

d. [2 points] Does the series $\sum_{n=1}^{\infty} \sin(a_n)$ converge?

$\boxed{\text{Converge}}$ Diverge ~~Cannot determine~~

Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$, and since convergence of $\sum_{n=1}^{\infty} a_n$ implies that $\lim_{n\to\infty} a_n = 0$ by n^{th} term test, $\therefore$ $0 < a_n < \pi$ for all big n, so $b_n := \sin(a_n)$ is positive for big n and $\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{x\to 0} \frac{\sin x}{x} = 1$, whence $\sum_{n=1}^{\infty} b_n$ converges by LCT.

e. [2 points] Does the series $\sum_{n=1}^{\infty} (a_n + b_n)^2$ converge?

Converge Diverge $\boxed{\text{Cannot determine}}$

Examples: converges when $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$, diverges when $b_n = 1$.

f. [2 points] Does the series $\sum_{n=1}^{\infty} e^{-b_n}$ converge?

Converge $\boxed{\text{Diverge}}$ Cannot determine

$e^{-b_n} = \frac{1}{e^{b_n}} \ge \frac{1}{e^M}$, so $\sum_{n=1}^{\infty} e^{-b_n}$ diverges by n^{th} term test.

5 ~~2~~. [10 points] Consider an outdoor pool initially filled with 20,000 gallons of water. Each day 4% of the water in the pool evaporates. Each morning at 10:00am, W gallons of water are added back to the pool where W is a constant.

a. [3 points] Let A_n be the number of gallons of water in the pool immediately after water is added back to the pool for the n^{th} time. Given that $A_1 = 19200 + W$, find A_2 and A_3. Put your final answers in the answer blanks.

$A_2 =$ $(0.96)^2 \cdot 20000 + W \cdot \frac{(0.96)^2 - 1}{0.96 - 1}$

$A_3 =$ $(0.96)^3 \cdot 20000 + W \cdot \frac{(0.96)^3 - 1}{0.96 - 1}$

b. [4 points] Find a closed form expression for A_n (i.e. evaluate any sums and solve any recursion). Note your answer may contain the constant W.

Since $A_0 = 20{,}000$

and $A_n = 0.96 A_{n-1} + W$,

we get

$$A_n = (0.96)^n A_0 + W \cdot \frac{(0.96)^n - 1}{(0.96) - 1}$$

$$= \boxed{(0.96)^n \cdot 20000 + W \cdot \frac{(0.96)^n - 1}{0.96 - 1}}$$

c. [3 points] If the pool has a maximum capacity of 25,000 gallons, find the largest value of W so that the pool does not overflow eventually.

We need $(0.96)^n \cdot 20000 + W \cdot \frac{(0.96)^n - 1}{0.96 - 1} \le 25000$

for every n, or equivalently

$$W \le \frac{(25000 - (0.96)^n \cdot 20000)}{\frac{(0.96)^n - 1}{0.96 - 1}}.$$

When $n \to \infty$, the right side $\to 25000 \cdot (1 - 0.96) = 1000$,

so we must have $W \le 1000$.

When $W = 1000$ we have $A_n = (0.96)^n \cdot (-5000) + 25000$ which is < 25000.

So answer is $W = 1000$.