Review Sheet for Final Exam (Math 156)

1. **True or false**

a) TRUE. It equals $\int_0^1 (1+x)dx = \left(x + \frac{x^2}{2}\right)\Big|_0^1 = \frac{3}{2}$ (change $\frac{i}{n} \to x$, $\frac{1}{n} \to dx$)

b) TRUE. It equals $\int_0^1 \frac{1}{1+x}dx = \ln(1+x)|_0^1 = \ln 2$

c) TRUE. It equals $\int_a^b f'(x)dx = f(b) - f(a)$ by FTC.

d) TRUE. Since $\int_0^1 f(x)g''(x)dx = \int_0^1 f(x)dg'(x) = f(x)g'(x)|_0^1 - \int_0^1 g'(x)df(x) = f(1)g'(1) - f(0)g'(1) - \int_0^1 f'(x)g'(x)dx = -\int_0^1 f'(x)g'(x)dx$ and $\int_0^1 g(x)f''(x)dx = \int_0^1 g(x)df'(x) = g(x)f'(x)|_0^1 - \int_0^1 f'(x)dg(x) = g(1)f'(1) - g(0)f'(1) - \int_0^1 g'(x)f'(x)dx = -\int_0^1 f'(x)g'(x)dx$, left hand side equals right hand side.

e) FALSE. Since $\int_0^1 \frac{dx}{x^2}$ diverges by $p-$test $\implies$ $\int_0^\infty \frac{dx}{x^2}$ diverges.

f) FALSE. The work done for spring $W = \int_{10}^{15} k(x-10)dx = \int_0^5 kxdx = \frac{1}{2}kx^2\Big|_0^5 = 200$ (N$\cdot$ cm) $\implies$ $k = 16$. $W = \int_{10}^{20} k(x-10)dx = \int_0^{10} kxdx = \frac{1}{2}kx^2\Big|_0^{10} = 800$ (N$\cdot$ cm)=8 Joule. (The general rule is: both starting from the natural length, if the length stretched is doubled, the work is multiplied by 4.)

g) FALSE. Do not try to find the true CM, it is complicate, instead, draw a graph and think. $\bar{x} = 0$ is correct since the graph is symmetric about $x = 0$. But $\bar{y} = \frac{1}{2}$ is lower than the actual $\bar{y}$. If the region is a rectangle $-1 \le x \le 1$ and $0 \le y \le 1$, this CM is true, but the actual graph is well above $y = 1$ (the lowest point of $\cosh x$ is 1 at $x = 0$). Actually, the true $\bar{y} = 0.5985$.

h) FALSE. A counterexample is an exponential distribution, $f(x)$ attains its maximum value at $x = 0$ rather than $\mu = \frac{1}{c}$. (For normal distribution, the statement is true.)

i) FALSE. The statement describes a <u>linear</u> decay. But the radioactive material obeys <u>exponential</u> decay $\left(y(t) = y_0 e^{-kt}\right)$, after 100 year (half-life) only $\frac{1}{2}$kg left, after 400 year $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$ kg left.

j) TRUE. Compounded Continuously $y(t) = y_0 e^{rt} = y_0(1 + rt + \frac{(rt)^2}{2} + \text{Remainder})$, where $t = 2$, $r = 0.05$ $\implies$ $y(2) = 2000 \cdot (1 + 0.05 \cdot 2 + \frac{1}{2}(0.05 \cdot 2)^2) + \text{Remainder} = 2210 + \text{Remainder} > 2210$ (The Remainder is positive.)

k) FALSE. It depends on the stability of the constant solution, if c is stable, the statement is true, if c is unstable, the statement is false.

l) TRUE. Geometric series $= \frac{1}{1-\frac{2008}{2009}} = \frac{2009}{2009-2008} = 2009$.

m) FALSE. A counterexample is $a_n = \frac{1}{n}$, $b_n = n$, $\lim_{n\to\infty} a_n b_n = 1$.

n) FALSE. A counterexample is $a_n = \frac{1}{n^2+1}$, $b_n = \frac{1}{n+1}$. $\sum_{n=0}^{\infty} a_n$ converges, while $\sum_{n=0}^{\infty} b_n$ diverges.

o) FALSE. A counterexample is $a_n = \frac{1}{n+1}$. $a_n \to 0$ as $n \to \infty$, but $\sum_{n=0}^{\infty} a_n$ diverges.

p) FALSE. A counterexample is $a_n = (-1)^n \frac{1}{n+1}$. $\sum_{n=0}^{\infty} a_n$ converges by AST, but $\sum_{n=0}^{\infty} |a_n|$ diverges.

q) FALSE. Since $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$, the ratio test in inconclusive.

r) TRUE. Because the interval of convergence (ioc) is centered at $a = 0$ in this case, if the power series converges for $x = 2$, the ioc must cover $-2 < x \le 2$, $x = 1$ is in the ioc.

s) FALSE. Do not try to find $f^{(3)}(0)$ and $f^{(6)}(0)$ directly, instead, using the Taylor series to derive them, since the general form for Taylor series $f(x) = \sum_{n=0}^{\infty} c_n x^n$ where $c_n = \frac{f^{(n)}(0)}{n!} \implies f^{(n)}(0) = n! \cdot c_n$. Since $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$, $f(x) = e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!}(-x^2)^n = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \text{Remainder}$. Here $c_3 = 0$ since there is not x^3 term, $c_6 = -\frac{1}{6}$, thus $f^{(3)}(0) = 0$, $f^{(6)}(0) = 6! \cdot c_6 = 720 \cdot \left(-\frac{1}{6}\right) = -120$. Thus the statement is false.

t) TRUE. Since $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-1)^n x^n$. Differentiate on both side yields $-\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty}(-1)^n n x^{n-1} \implies \frac{1}{(1+x)^2} = \sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1}$

u) TRUE. Since $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$, let $x = 1$, $e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} \cdots$, it is clear than $e > 2$. Consider a series $1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = 1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{1-\frac{1}{2}} = 3$, this series is greater than the series of e, since except the first three terms ($1 = 1, 1 = 1, \frac{1}{2} = \frac{1}{2}$), each term in this series is greater than the corresponding term in the series of e, $\frac{1}{4} > \frac{1}{3!} = \frac{1}{6}$, $\frac{1}{2^3} = \frac{1}{8} > \frac{1}{4!} = \frac{1}{24}$, $\cdots$ if $\frac{1}{2^{n-1}} > \frac{1}{n!}$, then $\frac{1}{2^{n-1} \cdot 2} = \frac{1}{2^{n+1}} > \frac{1}{n! \cdot (n+1)} = \frac{1}{(n+1)!}$, since $n + 1 > 2$ for $n \ge 1$, by induction, the series of 3 is greater than the series of e, since each term in the former is greater (or equal) the corresponding term in the latter. Thus $2 < e < 3$.

v) TRUE. Recall 1(s), $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$, $f(x) = e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!}(-x^2)^n = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \text{Remainder}$ $\implies e^{-x^2} > 1 - x^2$ since the next term $\frac{1}{2}x^4$ is positive, one can show that the remainder is positive (omit here). Thus $\int_0^1 e^{-x^2} dx > \int_0^1 (1 - x^2) dx = \left(x - \frac{x^3}{3}\right)\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$.

w) TRUE. Since the series is $\sin\frac{\pi}{2}$ (note that $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$) and $\sin\frac{\pi}{2} = 1$.

x) FALSE. Two ways to show this 1) by L'Hospital rule; 2) by Taylor series, ie substitute Taylor series of $\sin x$.

$\lim\limits_{x\to 0}\frac{\sin x}{x} = \lim\limits_{x\to 0}\frac{(\sin x)'}{(x)'} = \lim\limits_{x\to 0}\frac{\cos x}{1} = \cos 0 = 1.$

y) TRUE. Two ways to show this 1) by L'Hospital rule; 2) by Taylor series, ie., substitute Taylor series of $\cos x$.

$\lim\limits_{x\to 0}\frac{1-\cos x}{x^2} = \lim\limits_{x\to 0}\frac{(1-\cos x)'}{(x^2)'} = \lim\limits_{x\to 0}\frac{\sin x}{2x} = \frac{1}{2}.$

z) FALSE. $\lim\limits_{n\to\infty}\left(1-\frac{x}{n}\right)^n = \lim\limits_{n\to\infty}\left(1+\frac{-x}{n}\right)^n = e^{-x} \neq -e^x$

aa) TRUE. $\cosh^2 x - \sinh^2 x = \left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2 = \frac{e^{2x}+2+e^{-2x}}{4} - \frac{e^{2x}-2+e^{-2x}}{4} = 1.$

bb) FALSE. $\int \tanh x dx = \int \frac{\sinh x}{\cosh x}dx = \int \frac{1}{\cosh x}d\cosh x = \ln(\cosh x) \neq \text{sech}^2 h$. Note that $(\tanh x)' = \text{sech}^2 x$.

cc) TRUE. Since $(\sinh x)' = \cosh x$ and $\sqrt{1+\sinh^2 x} = \sqrt{\cosh^2 x} = \cosh x$, thus $\sinh x$ satisfies the equation.

dd) FALSE. Using the binomial series $(1+x)^k = 1+kx+\cdots$, replace x with $x^2 \implies (1+x^2)^k = 1+kx^2+\cdots$, since $k=\frac{1}{2}$, $\sqrt{1+x^2} = (1+x^2)^{\frac{1}{2}} = 1+\frac{1}{2}x^2+\cdots$.

ee) TRUE. Since $\cosh ix = \frac{e^{ix}+e^{-ix}}{2} = \frac{\cos x + i\,\sin x + \cos x - i\,\sin x}{2} = \cos x$

ff) TRUE. Since $e^{\pi i} = \cos\pi + i\,\sin\pi = -1 \implies \pi i = \log(-1)$. (Actually, more rigorous $\log(-1) = (2k+1)\pi i$, where k is an integer.)

gg) FALSE. $\binom{6}{3} = \frac{6!}{3!3!} = \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1} = 20$

hh) TRUE. $\binom{10}{2} = \frac{10!}{2!\cdot 8!} = \frac{10!}{8!2!} = \binom{10}{8}$.

integration

Question 2 Solution

Note that all these problems are $\frac{0}{0}$-type.

a) By L'Hospital rule $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \lim\limits_{x\to 0}\frac{(\sqrt{1+x}-1)'}{x'} = \lim\limits_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$

b) $\lim\limits_{h\to 0}\frac{(x+h)^4-x^4}{h} = \lim\limits_{h\to 0}\frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4-x^4}{h} = \lim\limits_{h\to 0}\frac{4x^3h+6x^2h^2+4xh^3+h^4}{h} = \lim\limits_{h\to 0}(4x^3+6x^2h+4xh^2+h^3) = 4x^3$ (one can use L'hospital rule, note that 'h' is variable here, regard x as constant)

c) $\lim\limits_{h\to 0}\frac{\int_0^h f(x)dx}{h} \xrightarrow{\text{L'Hospital Rule}} \lim\limits_{h\to 0}\frac{(\int_0^h f(x)dx)'}{(h)'} = \lim\limits_{h\to 0}\frac{f(h)}{1} = f(0)$

d) $\lim\limits_{h\to 0}\frac{\int_0^h xf(x)dx}{h^2} \xrightarrow{\text{L'Hospital Rule}} \lim\limits_{h\to 0}\frac{(\int_0^h xf(x)dx)'}{(h^2)'} = \lim\limits_{h\to 0}\frac{hf(h)}{2h} = \lim\limits_{h\to 0}\frac{f(h)}{2} = \frac{f(0)}{2}$

Question 3 Solution

a) $\int \frac{dx}{4x^2} = -\frac{1}{4x} + C$

b) $\int \frac{x}{4+x^2}dx = \int \frac{\frac{1}{2}}{4+x^2}dx^2 = \int \frac{\frac{1}{2}}{4+x^2}d(4+x^2) = \frac{1}{2}\ln(4+x^2) + C$

If you do not like the above way, using variable change $u = 4 + x^2$, $du = 2xdx \implies dx = \frac{1}{2x}du$

$\int \frac{x}{4+x^2}dx = \int \frac{x}{u} \cdot \frac{1}{2x}du = \int \frac{1}{2u}du = \frac{1}{2}\ln u + C = \frac{1}{2}\ln(4+x^2) + C$

c) This type of antiderivative, one needs to use substitution $x = 2\tan\theta$, $dx = 2(1+\tan^2\theta)d\theta$

$\int \frac{dx}{4+x^2} = \int \frac{2(1+\tan^2\theta)}{4+4\tan^2\theta}d\theta = \int \frac{1}{2}d\theta = \frac{1}{2}\theta + C = \frac{1}{2}\arctan\frac{x}{2} + C$

Write down this formula $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a}$

d) This type of antiderivative, one needs to use the formula $1 + \sinh^2\theta = \cosh^2\theta$, using variable change $x = 2\sinh\theta$, $dx = 2\cosh\theta d\theta$,

$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\cosh\theta}{\sqrt{4+4\sinh^2\theta}}d\theta = \int \frac{2\cosh\theta}{\sqrt{4\cosh^2\theta}}d\theta = \int \frac{2\cosh\theta}{2\cosh\theta}d\theta = \int d\theta = \theta + C = \text{arcsinh}\frac{x}{2} + C = \sinh^{-1}\frac{x}{2} + C$

Write down this formula $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a} + C$

e) Using partial fraction

$\int \frac{dx}{4-x^2} = \int \frac{dx}{(2+x)(2-x)} = \int \left(\frac{1/4}{2+x} + \frac{1/4}{2-x}\right)dx = \int \frac{1/4}{2+x}dx + \int \frac{1/4}{2-x}dx = \frac{1}{4}\ln|2+x| - \frac{1}{4}\ln|2-x| + C$

f) partial fraction again

$\int \frac{dx}{4x-x^2} = \int \frac{dx}{x(4-x)} = \int \left(\frac{1/4}{x} + \frac{1/4}{4-x}\right)dx = \int \frac{1/4}{x}dx + \int \frac{1/4}{4-x}dx = \frac{1}{4}\ln|x| - \frac{1}{4}\ln|4-x| + C$

g) integration by parts

$\int x\sin xdx = \int x(-1)d\cos x = -\int xd\cos x = -x\cos x + \int \cos xdx = -x\cos x + \sin x + C = \sin x - x\cos x + C$

h) using integration by parts twice

Using integration once

$\int e^{-x}\sin xdx = \int e^{-x}(-1)d\cos x = -e^{-x}\cos x + \int \cos xde^{-x} = -e^{-x}\cos x - \int e^{-x}\cos xdx$

Using integration twice

$\int e^{-x}\cos xdx = \int e^{-x}d\sin x = e^{-x}\sin x - \int \sin xde^{-x} = e^{-x}\sin x + \int e^{-x}\sin xdx$

Thus

$\int e^{-x}\sin xdx = -e^{-x}\cos x - e^{-x}\sin x - \int e^{-x}\sin xdx \implies 2\int e^{-x}\sin x = -e^{-x}\cos x - e^{-x}\sin x$

$\implies \int e^{-x}\sin x = \frac{1}{2}\left(-e^{-x}\cos x - e^{-x}\sin x\right) + C = -\frac{1}{2}e^{-x}(\cos x + \sin x) + C$

i) using integration by parts once and using the equality $\sin^2 x + \cos^2 x = 1$

$\int \sin^2 xdx = \int \sin x \cdot \sin xdx = \int \sin x(-1)d\cos x = -\sin x \cdot \cos x + \int \cos xd\sin x = -\sin x \cdot \cos x + \int \cos^2 xdx = -\sin x \cdot \cos x + \int (1 - \sin^2 x)dx = -\sin x \cdot \cos x + \int 1dx - \int \sin^2 xdx \implies$

$2\int \sin^2 xdx = -\sin x \cdot \cos x + x + C \implies \int \sin^2 xdx = -\frac{1}{2}\sin x\cos x + \frac{x}{2} + C = \frac{x}{2} - \frac{1}{4}\sin 2x + C$

<u>A simple way</u>, using $\sin^2 x = \frac{1-\cos 2x}{2}$, $\sin 2x = 2\cos x\sin x$, $(\cos 2x)^2 = (1+\cos 4x)/2$

Thus $\int \sin^2 xdx = \int \frac{1-\cos 2x}{2}dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$

j) using the equality $\sin^2 x + \cos^2 x = 1$

$\int \sin^3 xdx = \int \sin^2 x \cdot \sin xdx = \int (1-\cos^2 x)(-1)d\cos x = -\int (1-\cos^2 x)d\cos x = -\int 1d\cos x +$

$\int \cos^2 xd\cos x = -\cos x + \frac{1}{3}\cos^3 x + C$

k) using integration by parts will be too complicated, using $\sin^2 x = \frac{1-\cos 2x}{2}$, $\sin 2x = 2\cos x\sin x$,

$(\cos 2x)^2 = (1+\cos 4x)/2$

Thus $\int \sin^4 x = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \int \frac{1-2\cos 2x+\cos^2 2x}{4}dx = \int \frac{1-2\cos 2x+1-\sin^2 2x}{4}dx$

$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x - \frac{1}{4}\sin^2 2x\right)dx = \frac{x}{2} - \frac{1}{4}\sin 2x - \frac{1}{8}\int \sin^2 2xd(2x)$

$\xrightarrow{using\ 3\ i)x\to 2x} \frac{x}{2} - \frac{1}{4}\sin 2x - \frac{1}{8}\left(\frac{2x}{2} - \frac{\sin 4x}{4}\right) = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

<u>Question 4 Solution</u>

Using variable substitution $u = \frac{\pi}{2} - x$, $du = -dx$

$\int_{\pi/2}^{0} \frac{\sin(\frac{\pi}{2}-u)}{\sin(\frac{\pi}{2}-u)+\cos(\frac{\pi}{2}-u)}(-1)du = -\int_{\pi/2}^{0} \frac{\cos u}{\cos u+\sin u}du = \int_0^{\pi/2} \frac{\cos u}{\cos u+\sin u}du$

both u and x are integral variables changing from 0 to $\frac{1}{2}$, one may replace them with θ, thus $\int_0^{\pi/2} \frac{\sin\theta}{\sin\theta+\cos\theta}d\theta = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta+\cos\theta}d\theta$ on one hand, one the other hand $\int_0^{\pi/2} \frac{\sin\theta}{\sin\theta+\cos\theta}d\theta + \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta+\cos\theta}d\theta =$ $\int_0^{\pi/2} \left(\frac{\sin\theta}{\sin\theta+\cos theta} + \frac{\cos\theta}{\sin\theta+\cos\theta}\right)d\theta = \int_0^{\pi/2} 1d\theta = \frac{\pi}{2}$.

Thus $\int_0^{\pi/2} \frac{\sin\theta}{\sin\theta+\cos\theta}d\theta = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta+\cos\theta}d\theta = \frac{\pi}{4}$

<u>Question 5 Solution</u>

a) convergent by $p-$test $\int_1^\infty \frac{1}{x^2}dx = \int_0^\infty \left(-\frac{1}{x}\right)' dx = -\frac{1}{x}\Big|_1^\infty = 1$

b) divergent by $p-$test $\int_1^\infty \frac{dx}{x} = \ln x|_1^\infty = \infty$

c) $\int_1^\infty \frac{dx}{x-1} = \int_0^\infty \frac{dy}{y}$ divergent by $p-$test

d) divergent by p-test $\int_0^1 \frac{dx}{x^2} = -\frac{1}{x}\Big|_0^1 = \infty$

e) convergent by p-test $\int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}|_0^1 = 2$

f) divergent by p-test Since both $\int_{-1}^0 \frac{dx}{x}$ and $\int_0^1 \frac{dx}{x}$ diverges.

<u>Question 6 Solution</u>

Using substitution $r^2 - 2rx + a^2 = y$, $-2rdx = dy$, since x changes from $-a$ to a, y changes from

$r^2 - 2r(-a) + a^2 = r^2 + 2ra + a^2 = (r+a)^2$ to $r^2 - 2ra + a^2 = (r-a)^2$

$V(r) = \frac{q}{2a}\int_{-a}^{a} \frac{dx}{\sqrt{r^2-2rx+a^2}} \xrightarrow{x \to y} \frac{q}{2a}\int_{(r+a)^2}^{(r-a)^2} \frac{-\frac{1}{2r}dy}{\sqrt{y}} = -\frac{q}{4ar}\cdot 2\sqrt{y}\Big|_{(r+a)^2}^{(r-a)^2}$

$= -\frac{q}{2ar}\left(|r-a| - |r+a|\right) = \begin{cases} \frac{q}{r} \text{ if } r \geq a \\ \frac{q}{a} \text{ if } 0 \leq r \leq a \end{cases}$

<u>Question 7 Solution</u>

Divide the water into many layers, each is a rectangle with length l, width w, and height dz, put the origin ($z=0$) on the top layer (downward), the work done is $\int_0^h g \cdot l \cdot w \cdot z dz = g \cdot l \cdot w \cdot \frac{1}{2}z^2\Big|_0^h = \frac{1}{2}glwh^2$ (where g is the acceleration due to gravity). Actually the center of the mass is on the level $\frac{h}{2}$, the total mass is lwh, assume $\rho = 1$, pumping the water out the top is equivalent to move the center of the mass from $z = \frac{h}{2}$ to the top $z=0$, thus $\frac{1}{2}glwh^2$.

Substitute values, the work done is $\frac{1}{2}g \cdot 2 \cdot 1 \cdot 0.5^2 = \frac{1}{4}g$

From the formula $\frac{1}{2}glwh^2$, it is clear that if the width is double, the work is also doubled. If the height is doubled, the work is multiplied by 4.

<u>Question 8 Solution</u>

a) On x-axis, since one ion is held fixed at $x=0$, the distance is x, replace r in $F = -\frac{q^2}{r^2}$ with x, Work $= \int F(x)dx = \int_3^2 -\frac{q^2}{x^2}dx = \frac{q^2}{x}\Big|_3^2 = \frac{q^2}{2} - \frac{q^2}{3} = \frac{q^2}{6}$

b) On x-axis, since one ion is held fixed at $x=1$, the distance <u>becomes</u> $x-1$, replace r in $F = -\frac{q^2}{r^2}$ with $x-1$, Work $= \int F(x)dx = \int_3^2 -\frac{q^2}{(x-1)^2}dx = \frac{q^2}{x-1}\Big|_3^2 = \frac{q^2}{2-1} - \frac{q^2}{3-1} = \frac{q^2}{2}$

c) Add the results in a) and b) together Work$=\frac{q^2}{6} + \frac{q^2}{2} = \frac{2}{3}q^2$. ie, the work can be calculated with respect to A($x=0$) and B($x=1$), respectively, then put together.

d) Divide the rod into many small pieces, each has width Δw, here we use w to denote the position of small pieces (w changes from 0 to 1, it overlaps $x=0$ to $x=1$), for each piece the charge is $q\Delta w$, and the force $F(x) = -\frac{q \cdot q\Delta w}{(x-w)^2}$, work contributed by each piece $= \int_3^2 F(x)dx = \int_3^2 -\frac{q^2\Delta W}{(x-w)^2}dx = \frac{q^2\Delta w}{x-w}\Big|_3^2 = \left(\frac{1}{2-w} - \frac{1}{3-w}\right)q^2\Delta w$. Then we need a second integral for w from 0 to 1 to sum all the pieces, Total Work$= \int_0^1 \left(\frac{1}{2-w} - \frac{1}{3-w}\right)q^2\Delta w \xrightarrow{\Delta w \to dw} \int_0^1 \left(\frac{1}{2-w} - \frac{1}{3-w}\right)q^2 dw = q^2\left[-\ln(2-w) + \ln(3-w)\right]\Big|_0^1 = q^2 \ln\frac{4}{3} \approx 0.28q^2$ (this result is reasonable since it is larger than $\frac{q^2}{6} \approx 0.16q^2$ (case a) and smaller than $\frac{q^2}{2} \approx 0.5q^2$ (case b), cases a and b are two extreme cases (if we put all charge to one end of the rod), given that in the three cases (a,b,d) the total charge is the same.

Question 9 Solution

$f(x) = \cosh x$

a) arclength $= \int_{-1}^{1} \sqrt{1+[f'(x)]^2}dx = \int_{-1}^{1} \sqrt{1+[\cosh(x)']^2}dx = \int_{-1}^{1} \sqrt{1+\sinh^2 x}dx = \int_{-1}^{1} \sqrt{\cosh^2 x}dx$
$= \int_{-1}^{1} \cosh x dx = \sinh x|_{-1}^{1} = \sinh 1 - \sinh(-1) = 2\sinh 1$

b) surface $= \int_{-1}^{1} 2\pi f(x)\sqrt{1+[cosh(x)']^2}dx = 2\pi \int_{-1}^{1} \cosh^2 x dx = 2\pi \int_{-1}^{1} \left(\frac{e^x+e^{-x}}{2}\right)^2 dx = \frac{2\pi}{4}(e^2 - e^{-2}+4)$

($\int \cosh^2 x dx = \frac{1}{2}\cosh x \sinh x + \frac{1}{2}x$ see review problem for 2nd midterm exam.)

Question 10 Solution

$m = \int_a^b f(x)dx$, $\bar{x} = \frac{1}{m}\int xf(x)dx$, $\bar{y} = \frac{1}{m}\int \frac{1}{2}f^2(x)dx$ in (a,c,d)

or $m = \int_a^b [f(x)-g(x)]dx$, $\bar{x} = \frac{1}{m}\int x[f(x)-g(x)]dx$, $\bar{y} = \frac{1}{m}\int \frac{1}{2}[f^2(x)-g^2(x)]dx$ in (b)

$(\bar{x},\bar{y}) =$ (a) $\left(\frac{3}{2},\frac{6}{5}\right)$ (b) $\left(\frac{3}{4},\frac{12}{5}\right)$ (c) $\left(0,\frac{2}{5}\right)$ (d) $\left(\infty,\frac{1}{4}\right)$

(d) $m = \int_0^\infty \frac{1}{1+x^2}dx$ ($= \arctan x|_0^1 = \frac{\pi}{2}$)

variable change $x = \tan\theta$, $dx = (1+\tan^2\theta)d\theta$, x changes from 0 to ∞, corresponding to θ changes from 0 to $\frac{\pi}{2}$ since $x=\tan\theta$, $\tan 0 = 0$, $\tan\frac{\pi}{2} = \infty$

$$m = \int_0^\infty \frac{1}{1+x^2}dx = \int_0^{\frac{\pi}{2}} \frac{1+\tan^2\theta}{1+\tan^2\theta}d\theta = \int_0^{\frac{\pi}{2}} 1 d\theta = \arctan x|_0^1 = \frac{\pi}{2}$$

$$\bar{x} = \frac{\int_0^\infty xf(x)dx}{m} = \frac{\int_0^\infty \frac{x}{1+x^2}dx}{m} = \frac{\int_0^\infty \frac{\frac{1}{2}dx^2}{1+x^2}}{\frac{\pi}{2}} = \frac{\frac{1}{2}\ln(1+x^2)|_0^\infty}{\frac{\pi}{2}} = \infty \ \ !!!$$

The area is finite, it has a infinitely large $\bar{x}$

$$\bar{y} = \frac{\int_0^\infty \frac{1}{2}f^2(x)dx}{m} = \frac{1}{m}\cdot\int_0^\infty \frac{1}{2}f^2(x)dx = \frac{1}{\frac{\pi}{2}}\int_0^\infty \frac{1}{2}\left(\frac{1}{1+x^2}\right)^2 dx = \frac{2}{\pi}\int_0^\infty \frac{\frac{1}{2}}{(1+x^2)^2}dx = \frac{2}{\pi}\int_0^\infty \frac{\frac{1}{2}+\frac{1}{2}x^2-\frac{1}{2}x^2}{(1+x^2)^2}dx$$
$$= \frac{2}{\pi}\left(\int_0^\infty \frac{\frac{1}{2}(1+x^2)}{(1+x^2)^2}dx - \int_0^\infty \frac{\frac{1}{2}x^2}{(1+x^2)^2}dx\right) = \frac{2}{\pi}\left(\int_0^\infty \frac{\frac{1}{2}}{1+x^2}dx - \int_0^\infty \frac{\frac{1}{2}x\cdot x}{(1+x)^2}dx\right)$$
$$= \frac{2}{\pi}\left(\frac{1}{2}\arctan x|_0^\infty - \int_0^\infty \frac{\frac{1}{2}x\cdot\frac{1}{2}dx^2}{(1+x^2)^2}\right) = \frac{2}{\pi}\left[\frac{1}{2}\cdot\frac{\pi}{2} + \int_0^\infty \frac{x}{4}\,d\left(\frac{1}{1+x^2}\right)\right] = \frac{2}{\pi}\left(\frac{\pi}{4} + \frac{x}{4}\cdot\frac{1}{1+x^2}\Big|_0^\infty - \frac{1}{4}\int_0^\infty \frac{1}{1+x^2}dx\right)$$
$$= \frac{2}{\pi}\left(\frac{\pi}{4} + \frac{x}{4}\cdot\frac{1}{1+x^2}\Big|_0^\infty - \frac{1}{4}\int_0^\infty \frac{1}{1+x^2}dx\right) = \frac{2}{\pi}\left(\frac{\pi}{4} + \frac{x}{4}\cdot\frac{1}{1+x^2}\Big|_0^\infty - \frac{1}{4}\arctan x|_0^\infty\right)$$
$$= \frac{2}{\pi}\left(\frac{\pi}{4} + 0 - \frac{1}{4}\cdot\frac{\pi}{2}\right) = \frac{2}{\pi}\cdot\frac{\pi}{8} = \frac{1}{4}$$

Question 11 Solution

$f(t) = ce^{-ct}$, where $c = \frac{1}{1000}$ and $t \geq 0$

a) Prob$(0 \leq t \leq 200) = \int_0^{200} ce^{-ct}dt = -e^{-ct}|_0^{200} = 1 - e^{-\frac{1}{5}} \approx 0.18$

b) Prob$(t \geq 800) = \int_{800}^\infty ce^{-ct}dt = -e^{-ct}|_{800}^\infty = e^{-\frac{4}{5}} \approx 0.45$

Question 12 Solution

We need to show that $\int_0^1 f(x)dx = 1$

$$\int_0^1 f(x)dx = \int_0^1 \frac{1}{\pi\sqrt{x(1-x)}}dx \xrightarrow{x=u+\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\pi\sqrt{(u+\frac{1}{2})(1-u-\frac{1}{2})}}du = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\pi\sqrt{(\frac{1}{2}+u)(\frac{1}{2}-u)}}du$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\pi\sqrt{\frac{1}{4}-u^2}}du \xrightarrow{u=\frac{1}{2}\sin\theta} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\pi\cdot\frac{1}{2}\cos\theta}\cdot\frac{1}{2}\cos\theta d\theta = \frac{1}{\pi}\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1$$

Differential Equations

<u>Question 13 Solution</u>

a) $y' = -2y$, $y_0 = 1$ (standard exponential decay) $\implies y = C\cdot e^{-2t}$.

$y_0 = 1 \implies C = 1 \implies y(t) = e^{-2t}$ and $\lim_{t\to\infty} y(t) = 0$

b) $y' = 1-2y \implies \frac{dy}{dt} = 1-2y$ separation of variables $\implies \frac{dy}{1-2y} = dt$ integrate both sides $\implies -\frac{1}{2}\ln|1-2y| = t + C \implies 1-2y = C\cdot e^{-2t} \implies y = \frac{1-C\cdot e^{-2t}}{2}$

$y_0 = 0 \implies C = 1 \implies y = \frac{1-e^{-2t}}{2}$ and $\lim_{t\to\infty} y(t) = \frac{1}{2}$

c) $y' = 1-y^2 \implies \frac{dy}{dt} = (1+y)(1-y)$ separation of variables $\implies \frac{dy}{(1+y)(1-y)} = dt$ partial fraction $\implies \frac{\frac{1}{2}}{1+y}dy + \frac{\frac{1}{2}}{1-y}dy = dt$ integrate both sides $\implies \frac{1}{2}\ln|1+y| - \frac{1}{2}\ln|1-y| = t + C$ $\implies \ln\left|\frac{1+y}{1-y}\right| = 2t + C \implies \frac{1+y}{1-y} = C\cdot e^{2t}$ where C is constant may be positive or negative $\implies y(t) = \frac{C\cdot e^{2t}-1}{C\cdot e^{2t}+1}$

$y_0 = 0 \implies C = 1 \implies y(t) = \frac{e^{2t}-1}{e^{2t}+1}$. Furthermore $y(t) = \frac{(e^{2t}-1)e^{-t}}{(e^{2t}+1)e^{-t}} = \frac{e^t-e^{-t}}{e^t+e^{-t}} = \tanh t$

Check... $y(t) = \tanh t$ is the solution.

$\lim_{t\to\infty} y(t) = 1$

d) $y' = -ty \implies \frac{dy}{dt} = -ty$ separation of variables $\implies \frac{dy}{y} = -tdt$ integrate both sides $\implies \ln|y| = -\frac{1}{2}t^2 + C \implies y = Ce^{-\frac{1}{2}t^2}$

$y_0 = 1 \implies C = 1 \implies y(t) = e^{-\frac{1}{2}t^2}$

$\lim_{t\to\infty} y(t) = 0$

<u>Question 14 Solution</u>

a) $y = c_1e^t + c_2e^{-t} \implies y' = c_1e^t - c_2e^{-t} \implies y'' = c_1e^t + c_2e^{-t} = y$, thus it is a solution of $y'' = y$ for any constants c_1, c_2.

b) $y(0) = 1, y'(0) = 0 \implies c_1 + c_2 = 1,\ c_1 - c_2 = 0 \implies c_1 = c_2 = \frac{1}{2} \implies y(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh x$

c) $y(0) = 0, y'(0) = 1 \implies c_1 + c_2 = 0,\ c_1 - c_2 = 1 \implies c_1 = \frac{1}{2}, c_2 = -\frac{1}{2} \implies y(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh x$

<u>Question 15 Solution</u>

$y(t) = y_0 e^{-kt}$

$y(t) = 40 \cdot \left(\frac{1}{2}\right)^{\frac{t}{1.4\times10^{-4}}}$

$30 = 40 \cdot \left(\frac{1}{2}\right)^{\frac{t}{1.4\times10^{-4}}}$

$t = 1.4 \times 10^{-4} \frac{\ln\frac{3}{4}}{\ln\frac{1}{2}} = 0.581 \times 10^{-4}\text{s}$

Question 16 Solution

$y' = \frac{2500-20y}{10000}$

$y' = \frac{1}{500}(125 - y)$ Newton's heating/cooling $y' = k(T - y)$

$y(t) = T + (y_0 - T)e^{-kt} = 125 + (y_0 - 125)e^{-\frac{t}{500}}$

approach to 125 kg.

Question 17 Solution

$y(t) = T + (y_0 - T)e^{-kt}$ Note that the patient's temperature is T, $y_0 = 70°\text{F}$

$$\begin{cases} 95 = T + (70 - T)e^{-k} \\ 100 = T + (70 - T)e^{-2k} \end{cases}$$

$\left(\frac{95-T}{70-T}\right)^2 = \frac{100-T}{70-T}$

$T = 101.25°$ F

Question 18 Solution

$y' = ky(M - y)$

$y(t) = \frac{My_0}{y_0+(M-y_0)e^{-kMt}}$

$y_0 = 10,\ M = 4000$

$y(t) = \frac{40000}{10+(4000-10)e^{-4000kt}}$

measure time in days

$20 = \frac{40000}{10+(4000-10)e^{-4000\cdot7k}}$

$e^{-k} = \left(\frac{199}{399}\right)^{\frac{1}{28000}}$

$y(t) = \frac{40000}{10+3990\left(\frac{199}{399}\right)^{\frac{t}{7}}}$

let $y(t) = \frac{1}{2} \cdot 4000 = 2000$, solve $t = \frac{7\ln 399}{\ln 399 - \ln 199} \approx 60$ days.

Series

Question 19 Solution

a) divergent $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ by $p - test$ of series, $p = 1$.

b) convergent since $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} = 1 < \infty$.

c) convergent by $p - test$ of series, $p = 2$.

d) convergent by Alternating Series Test, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$, $a_{n+1} < a_n$ and the sign is alternating.

e) divergent by Ratio Test, $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} = 2 > 1$, ($L > 1$ divergent)

<u>Question 20 Solution</u>

a) $0.111111.... = 0.1 + 0.01 + 0.001 + 0.0001 + \cdots = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{1000} + \cdots = \sum_{n=1}^{\infty} \frac{1}{10^n} = \frac{1}{10} \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n = \frac{1}{10} \cdot \frac{1}{1-\frac{1}{10}} = \frac{1}{9}$

b) $0.1212121212... = \frac{12}{100} + \frac{12}{10000} + \frac{12}{1000000} + \cdots = \sum_{n=1}^{\infty} \frac{12}{100^n} = \frac{12}{100} \sum_{n=0}^{\infty} \frac{1}{100^n} = \frac{12}{100} \cdot \frac{1}{1-\frac{1}{100}} = \frac{12}{99}$

c) $0.4999999... = 0.45 + 0.045 + 0.0045 + 0.00045 + \cdots = \frac{45}{100} + \frac{45}{1000} + \frac{45}{10000} + \cdots$

$= \frac{45}{100}\left(1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots\right) = \frac{45}{100} \sum_{n=0}^{\infty} \frac{1}{10^n} = \frac{45}{100} \cdot \frac{1}{1-\frac{1}{10}} = \frac{1}{2}$ <u>(ie, 0.49999999... = 0.5)</u>

<u>Question 21 Solution</u>

a) Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$, where $x = 2$.

b) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$ (telescoping series).

c) $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{3} \cdot \frac{1}{1-\frac{1}{3}} = \frac{1}{2}$ (note that n starts from 1)

d) $\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1}{3} \sum_{n=1}^{\infty} n \left(\frac{1}{3}\right)^{n-1} = \frac{1}{3} \sum_{n=1}^{\infty} (x^n)'\Big|_{x=\frac{1}{3}} = \frac{1}{3} \left(\sum_{n=0}^{\infty} x^n\right)'\Bigg|_{x=\frac{1}{3}} = \frac{1}{3} \cdot \left(\frac{1}{1-x}\right)'\Big|_{x=\frac{1}{3}} = \frac{1}{3} \cdot \frac{1}{(1-x)^2}\Big|_{x=\frac{1}{3}} = \frac{1}{3} \cdot \frac{1}{\left(1-\frac{1}{3}\right)^2} = \frac{3}{4}$

(Note that $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty}(n+1)x^n$)

e) $\sum_{n=1}^{\infty} \frac{1}{n3^n} = \sum_{n=1}^{\infty} \frac{x^n}{n}\Big|_{x=\frac{1}{3}} = \sum_{n=1}^{\infty} \int x^{n-1}dx\Big|_{x=\frac{1}{3}} = \int \sum_{n=1}^{\infty} x^{n-1}dx\Bigg|_{x=\frac{1}{3}} = \int \sum_{n=0}^{\infty} \int x^n dx\Bigg|_{x=\frac{1}{3}} = \int \frac{1}{1-x}dx\Big|_{x=\frac{1}{3}} = -\ln(1-x)\big|_{x=\frac{1}{3}} = \ln \frac{1}{1-x}\Big|_{x=\frac{1}{3}} = \ln \frac{3}{2}$

(Note that $\ln \frac{1}{1-x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}x^n$)

<u>Question 22 Solution</u>

Given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, note that $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$ includes all the odd terms. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ =odd terms+even terms $\implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\implies \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} + \frac{1}{4}\frac{\pi^2}{6} \implies \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}$.

<u>Question 23 Solution</u>

a) Use $|s - s_{10}| \leq \int_n^\infty f(x)dx$ since all terms are positive, where $f(x) = \frac{1}{x^2}$

$|s - s_{10}| \leq \int_{10}^\infty f(x)dx = \int_{10}^\infty \frac{1}{x^2}dx = -\frac{1}{x}\Big|_{10}^\infty = 0.1$

b) Use $|s - s_{10}| \leq a_{n+1}$ where $a_{n+1} = \frac{1}{(n+1)^2}$ since the series is an alternating series.

$|s - s_{10}| \leq a_{n+1} = \frac{1}{11^2} = \frac{1}{121}$

Question 24 Solution

Assume the dog starts with A running towards B, it will take $\frac{20}{10+2}$ hr to meet B, during this time interval A and B traveled $2 \cdot \frac{20}{10+2}$, respectively and the dog traveled $10 \cdot \frac{20}{10+2} = \frac{50}{3} = 25 \cdot \frac{2}{3}$. Then the dog will run from B towards A, the distance between A and B becomes $20 - 2 \cdot \frac{20}{10+2} - 2 \cdot \frac{20}{10+2} = \frac{40}{3}$. Everything is the same except 20 replaced by $\frac{40}{3}$, this time the dog will travel $\frac{10}{10+2} \cdot \frac{40}{3} = \frac{100}{9} = 25 \cdot \left(\frac{2}{3}\right)^2$

The series is $25 \cdot \frac{2}{3} + 25 \cdot \left(\frac{2}{3}\right)^2 + 25 \cdot \left(\frac{2}{3}\right)^3 + 25 \cdot \left(\frac{2}{3}\right)^4 + \cdots = 25 \cdot \frac{2}{3} \cdot \left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right) = 25 \cdot \frac{2}{3} \cdot \frac{1}{1-\frac{2}{3}} = 50$

The question requires to express D as an infinite series, actually a simple way to find the sum is that using distance = speed $\times$ time, where time $= \frac{20}{2+2} = 5$ hr, total time it will take for the two students to meet, then the distance $= 10 \times 5 = 50$ miles.

Question 25 Solution

Consider an arbitrary sequence of a finial win :

$1, -1, -1, -1, 1, -1, 1, -1, \cdots, 1, 1$

where -1 denotes lose, 1 denotes win. The sequence satisfies following properties:

1) It has even number of elements, denote the length of the sequence as $2n$ (n round), the sum equals 2, it is required that $a_{2n-1} = 1$ and $a_{2n} = 1$. The sequence must have a length of an even number. This is equivalent to, if you generate a sequence with -1 and 1, and the sum of the sequence is 2, the length of the sequence has to be an even number.

2) In this sequence, $a_{2i-1} = 1$ and $a_{2i} = 1$ ($i < n$) does not exist, otherwise, the game stops at i round rather than n rounds.

3) In this sequence, $a_{2i-1} = -1$ and $a_{2i} = -1$ ($i < n$) does not exist, otherwise, the game stops at i round rather than n rounds.

4) Thus in each round, there are two cases: eith er $\{a_{2i-1} = 1, a_{2i} = -1\}$ or $\{a_{2i-1} = -1, a_{2i} = 1\}$. In other words, in each round, a_{2i-1} and a_{2i} have opposite sign, for $i < n$. After each round

$(i < n)$, the score comes back to zero.

5) The possibility in each round $p(1-p)+(1-p)p = 2p(1-p)$.

6) The possibility for a sequence with a length $2n$ is $[2p(1-p)]^{n-1}p^2$.

The total possibility to a final win is

$$\sum_{n=1}^{\infty}[2p(1-p)]^{n-1}p^2 = \sum_{n=0}^{\infty}[2p(1-p)]^n p^2 = p^2\sum_{n=0}^{\infty}[2p(1-p)]^n = p^2\cdot\frac{1}{1-2p(1-p)} = \frac{p^2}{1-2p+2p^2}$$

$$p=\frac{1}{2} \implies \frac{p^2}{1-2p+2p^2}=\frac{1}{2}$$

$$p=\frac{1}{4} \implies \frac{p^2}{1-2p+2p^2}=\frac{1}{10}$$

$$p=\frac{3}{4} \implies \frac{p^2}{1-2p+2p^2}=\frac{9}{10}$$

<u>Question 26 Solution</u>

a) The total length removed $=\frac{1}{3}+2\cdot\frac{1}{3}\cdot\frac{1}{3}+4\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}+\cdots=\frac{1}{3}\left[1+\frac{2}{3}+\left(\frac{2}{3}\right)^2+\cdots\right]=\frac{1}{3}\cdot\frac{1}{1-\frac{2}{3}}=1$

b) The number of intervals is a sequence: $2, 4, 8, \cdots, 2^n, \cdots$, and $\lim_{n\to\infty} n^2=\infty$.

Power Series, Taylor Series

<u>Question 27 Solution</u>

a) $L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right|=|x|<1 \implies$ the radius of convergence is 1; since at two end points $x=\pm1$, the series diverges, the interval of convergence is $-1<x<1$. The sum is $\frac{1}{1-x}$ for $-1<x<1$.

b) $L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{2^n x^{n+1}}{2^{n+1}x^n}\right|=\left|\frac{x}{2}\right|<1 \implies |x|<2 \implies$ the radius of convergence is 2; since at two end points $x=\pm2$, the series diverges, the interval of convergence is $-2<x<2$. The sum is $\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n=\frac{1}{1-\frac{x}{2}}=\frac{2}{2-x}$ for $-2<x<2$.

c) $L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{(x-1)^n}\right|=|x-1|<1 \implies -1<x-1<1 \implies 0<x<2 \implies$ the radius of convergence is 1 (the length of the interval divided by 2); since at two end points $x=0$ and 2, the series diverges, the interval of convergence is $0<x<2$. The sum is $\frac{1}{1-(x-1)}=\frac{1}{2-x}$ for $0<x<2$.

d) $L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{nx^{n+1}}{(n+1)x^n}\right|=|x|<1 \implies -1<x<1 \implies$ the radius of convergence is 1 ; since at $x=1$, the series is harmonic series thus diverges, while at $x=-1$, the series converges by AST (alternating series test), the interval of convergence is $-1\le x<1$. Note that $x^n=\int nx^{n-1}dx \implies \frac{x^n}{n}=\int x^{n-1}dx$ the sum is $\sum_{n=1}^{\infty}\int x^{n-1}dx=\int\sum_{n=1}^{\infty}x^{n-1}dx=\int\sum_{n=0}^{\infty}\int x^n dx=\int\frac{1}{1-x}dx=-\ln(1-x)=\ln\frac{1}{1-x}$ for $-1\le x<1$

$\ln \frac{1}{1-x} = \sum\limits_{n=1}^{\infty} \frac{x^n}{n}$

e) $L = \lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{nx^n}\right| = |x| < 1 \implies -1 < x < 1 \implies$ the radius of convergence is 1; since at $x = \pm 1$, the series diverges, the interval of convergence is $-1 < x < 1$.

Note that $(x^n)' = nx^{n-1}$, $\sum\limits_{n=1}^{\infty} nx^n = x\sum\limits_{n=1}^{\infty} nx^{n-1} = x\sum\limits_{n=1}^{\infty} (x^n)' = x \cdot \left(\sum\limits_{n=0}^{\infty} x^n\right)' = x \cdot \left(\frac{1}{1-x}\right)' = x \cdot \frac{1}{(1-x)^2} = \frac{x}{(1-x)^2}$

$\frac{x}{(1-x)^2} = \sum\limits_{n=1}^{\infty} nx^n$

Question 28 Solution

Namely find c_n, such that $f(x) = \frac{1}{1-x} = \sum\limits_{n=0}^{\infty} c_n(x+1)^n$.

$\frac{1}{1-x} = \frac{1}{2-(x+1)} = \frac{1}{2} \cdot \frac{1}{1-\frac{x+1}{2}} = \frac{1}{2} \cdot \left[1 + \frac{x+1}{2} + \left(\frac{x+1}{2}\right)^2 + \left(\frac{x+1}{2}\right)^3 + \cdots\right] = \frac{1}{2} \cdot \sum\limits_{n=0}^{\infty} \left(\frac{x+1}{2}\right)^n = \sum\limits_{n=0}^{\infty} \frac{(x+1)^n}{2^{n+1}}$

$L = \lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\frac{2^{n+1}(x+1)^{n+1}}{2^{n+2}(x+1)^n}\right| = \left|\frac{x+1}{2}\right| < 1 \implies -2 < x+1 < 2 \implies -3 < x < 1$ the radius of convergence is 2 (the length of the interval divided by 2); since at $x = -3$ or $x = 1$, the series diverges, the interval of convergence is $-3 < x < 1$.

Set $x = \frac{1}{4}$, $\frac{4}{3} = \sum\limits_{n=0}^{\infty} \frac{\left(\frac{1}{4}+1\right)^n}{2^{n+1}} = \sum\limits_{n=0}^{\infty} \frac{1}{2}\left(\frac{5}{8}\right)^n = \frac{1}{2} + \frac{5}{16} + \frac{25}{128} + \frac{125}{1024} + \cdots$

Question 29 Solution

a) $a = 0$, $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum\limits_{n=0}^{\infty} (-x)^n$, the interval of convergence is $-1 < x < 1$ (by ratio test and consider the cases $x = \pm$)

b) $a = 1$, $\frac{1}{1+x} = \frac{1}{2-(1-x)} = \frac{1}{2} \cdot \frac{1}{-\frac{1-x}{2}} = \frac{1}{2}\sum\limits_{n=0}^{\infty} \left(\frac{1-x}{2}\right)^n = -\frac{1}{2}\sum\limits_{n=0}^{\infty} \left(\frac{x-1}{2}\right)^n$, this requires that $\left|\frac{x-1}{2}\right| < 1 \implies |x-1| < 2 \implies -2 < x-1 < 2 \implies -1 < x < 3 \implies$ the interval of converges. (For $x = -1$, $x = 3$, the series $\sum\limits_{n=0}^{\infty} (-1)^n$ and $\sum\limits_{n=0}^{\infty} 1$ both diverge.)

Question 30 Solution

$$f(x) = \sinh x \implies f(0) = \sinh 0 = 0$$
$$f'(x) = \cosh x \implies f'(0) = \cosh 0 = 1$$
$$f''(x) = \sinh x \implies f''(0) = \sinh 0 = 0$$
$$f'''(x) = \cosh x \implies f'''(0) = \cosh 0 = 1$$
$$\vdots$$

$\sinh x = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \cdots = \sum\limits_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$\cosh x = (\sinh x)' = \left(\sum\limits_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}\right)' = \sum\limits_{n=0}^{\infty} \left(\frac{x^{2n+1}}{(2n+1)!}\right)' = \sum\limits_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

Question 31 Solution

$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$\sin^2 x + \cos^2 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2 + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)^2$

$= \left(x^2 - 2\frac{x^4}{3!} + 2\frac{x^6}{5!} + \cdots\right) + \left(1 - 2\frac{x^2}{2} + \frac{x^4}{4} + 2\frac{x^4}{4!} + \cdots\right) = 1$

Question 32 Solution

Because $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{n!}$

$T_1(x) = 1$ and $T_2(x) = 1 - x^2$

Question 33 Solution

Show that $0 \leq f(x) < 1$, $\lim_{x\to\infty} f(x) = 1$, $\lim_{x\to 0^+} f^{(n)}(x) = P(\frac{1}{x})e^{-1/x}$, where $P(\frac{1}{x})$ is a polynomial of $\frac{1}{x}$, when $x \to 0^+$ $e^{-1/x} \to 0$ exponentially (faster than any polynomial) thus $f^{(n)}(x) \to 0$ regardless of the form of $P(\frac{1}{x})$.

Question 34 Solution

$\sqrt{x} = \sqrt{a} + \frac{1}{2\sqrt{a}}(x-a) - \frac{1}{8}a^{\frac{3}{2}}(x-a)^2 + Remainder$

set $a = 9$

$\sqrt{x} = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 + Remainder$

This is an alternating series, $|s - s_n| \leq a_{n+1}$, ie $\left|\sqrt{x} - \left(3 + \frac{1}{6}(x-9)\right)\right| \leq \frac{1}{216}(x-9)^2$

set $x = 10$

$\left|\sqrt{10} - 3.16666\right| \leq 0.00463 < 0.005$

The approximate value is 3.16666

Question 35 Solution

Since $f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$, an alternating series, thus $|s - s_n| \leq a_{n+1}$ where $a_{n+1} = \frac{x^{n+1}}{n+1}$, let $\frac{x^{n+1}}{n+1} = 10^{-3}$ (need to evaluate $\ln\frac{3}{2} = \ln(1+\frac{1}{2})$, ie, $x = \frac{1}{2}$), substitute $x = \frac{1}{2}$ yields $\frac{1}{2^{n+1}} = 0.001(n+1)$, test with $n = 1, 2, 3, 4, 5, 6, 7$, find that $n = 6$, roughly satisfies the equality. $n = 6$, $s_6 \approx 0.4047$, exact value $s \approx 0.4055$, error is within 10^{-3}

Question 36 Solution

The first two nonzero terms

a) $\tan x = x + \frac{x^3}{3} + \text{Remainder}$

b) $e^{-x}\sin x = x - x^2 + \text{Remainder}$

c) $\frac{1-\cos x}{x} = \frac{1}{2}x - \frac{1}{24}x^3 + \text{Remainder}$

Question 37 Solution

$f(x) = \frac{x}{e^x-1}$, $B_0 = f(0) = 1$ $B_1 = f'(0) = -\frac{1}{2}$ $B_1 = f''(0) = \frac{1}{6}$ (using L'Hospital rule).

Question 38 Solution

a) $f(x) = x, f(0) = 0, f'(0) = 1$

b) $f(x) = \sin x, f(0) = 0, f'(x) = \cos x, f'(0) = 1$

c) $f(x) = \ln(1+x), f(0) = 0, f'(x) = \frac{1}{1+x}, f'(0) = 1$

b) $f(x) = e^x - 1, f(0) = 0, f'(x) = e^x, f'(0) = 1$

If the functions are graphed in a neighborhood of $x = 0$, the order they appear (from top to bottom), consider their Taylor approximations

$x = x$, $\sin x = x - \frac{1}{6}x^3 + \cdots$, $\ln(1+x) = x - \frac{1}{2}x^2 + \cdots$, $e^x - 1 = x + \frac{1}{2}x^2$

Thus on right hand side of 0, from top to bottom, $e^x - 1$, x, $\sin x$ and $\ln(1+x)$; on left hand side of 0, from top to bottom, $e^x - 1$, $\sin x$, x, and $\ln(1+x)$.

Question 39 Solution

a) $J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots$ it is alternating series.

$\int_0^1 \left(1 - \frac{x^2}{4}\right) dx = \frac{11}{12}$

error bound $\int_0^1 \frac{x^4}{64} dx = \frac{x^5}{5 \cdot 64} = \frac{1}{320}$

b) $J_0(x)' = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)x^{2n+1}}{2^{2n+2}((n+1)!)^2}$

$J_0(x)'' = \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n-2}}{2^{2n}(n!)^2}$

$xJ_0(x)'' = \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n-1}}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)(2n+1)x^{2n+1}}{2^{2n+2}((n+1)!)^2}$

$xJ_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n}(n!)^2}$

$xJ_0(x)'' + J_0(x)' + xJ_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^{2n+1}}{2^{2n}(n!)^2}\left[\frac{2n+2}{2^2(n+1)^2} + \frac{(2n+1)(2n+2)}{2^2(n+1)^2} - 1\right] = 0$

Thus $J_0(x)$ satisfies $xy'' + y' + xy = 0$

Question 40 Solution

a) $f(t)' = \left(\sum_{n=0}^{\infty} t^n\right)' = 1 + t + 2t + 3t^2 + 4t^3 + \cdots = \sum_{n=0}^{\infty}(n+1)t^n$

$f^2(t) = \left(\sum_{n=0}^{\infty} t^n\right)^2 = 1 + t + 2t + 3t^2 + 4t^3 + \cdots = \sum_{n=0}^{\infty}(n+1)t^n$

Thus $f(t)' = f^2(t)$, $f(t)$ is solution of $y' = y^2$ and $f(0) = 1$

b) $y' = y^2 \implies \frac{dy}{dt} = y^2 \implies \frac{dy}{y^2} = dt \implies \int \frac{dy}{y^2} = \int dt \implies -\frac{1}{y} = t + C \implies y = \frac{1}{-C-t}$.
Substitute initial condition $t = 0, y = 1 \implies C = -1$, thus $y = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$ Geometric Series.

Question 41 Solution

a) $\int_0^\infty \frac{\sin x}{x} dx = \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x} dx = \int_0^\pi \frac{\sin x}{x} dx + \sum_{n=1}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x} dx \leq \int_0^\pi \frac{\sin x}{x} dx + \sum_{n=1}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{n\pi} dx$
since in each interval $x \geq n\pi$
$\int_0^\infty \frac{\sin x}{x} dx \leq \int_0^\pi \frac{\sin x}{x} dx + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_{n\pi}^{(n+1)\pi} \sin x dx = \int_0^\pi \frac{\sin x}{x} dx + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{-\cos x}{n} \Big|_{n\pi}^{(n+1)\pi} = \int_0^\pi \frac{\sin x}{x} dx + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi - \cos(n+1)\pi}{n} = \int_0^\pi \frac{\sin x}{x} dx + \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{2}{n} = \int_0^\pi \frac{\sin x}{x} dx + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges since the $n = 0$ term is finite (note that $\lim_{x \to 0} \frac{\sin x}{x} = 1$) and by AST.

b) $\int_0^\infty \left|\frac{\sin x}{x}\right| dx = \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x} dx = \int_0^\pi \frac{|\sin x|}{x} dx + \sum_{n=1}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x} dx \geq \int_0^\pi \frac{\sin x}{x} dx + \sum_{n=1}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{(n+1)\pi} dx$
since in each interval $x \leq n(+1)\pi$
$\int_0^\infty \frac{\sin x}{x} dx \geq \int_0^\pi \frac{\sin x}{x} dx + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n+1} \int_{n\pi}^{(n+1)\pi} |\sin x| \, dx = \int_0^\pi \frac{\sin x}{x} dx + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{2}{n+1} = \int_0^\pi \frac{\sin x}{x} dx + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n+1}$
diverges since the series is Harmonic Series.

Question 42 Solution

$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots$. It is an alternating series $|s - s_2| \leq a_3$

$\left|\sin \frac{\pi}{5} - \frac{\pi}{5} + \frac{1}{6} \cdot \left(\frac{\pi}{5}\right)^3\right| \leq \frac{1}{120} \left(\frac{\pi}{5}\right)^5$

$\left|\sin \frac{\pi}{5} - 0.586977\right| \leq 0.000816$

Question 43 Solution

a) $\frac{a}{a+b} = \frac{a}{b} \cdot \frac{1}{1+\frac{a}{b}} = \frac{a}{b} \cdot \frac{1}{1-\left(-\frac{a}{b}\right)} = \frac{a}{b} \cdot \sum_{n=0}^{\infty} \left(-\frac{a}{b}\right)^n$

$\frac{a}{a+b} = \frac{a}{b} \left(1 - \frac{a}{b} + \frac{a^2}{b^2} + \cdots\right) = \frac{a}{b} - \frac{a^2}{b^2} + \frac{a^3}{b^3} + \cdots$

b) using the Theorem

$(1+x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 + \cdots$ for $-1 < x < 1$

$\sqrt{R^2 - r^2} = R\sqrt{1 - \frac{r^2}{R^2}} = R\left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}} = R\left[1 - \frac{1}{2} \cdot \frac{r^2}{R^2} + \frac{(\frac{1}{2})(\frac{1}{2}-1)}{2}\left(-\frac{r^2}{R^2}\right)^2 + \cdots\right] = R - \frac{r}{2} \cdot \frac{r}{R} - \frac{r}{8} \cdot \frac{r^3}{R^3} + \cdots$

Question 44 Solution

Starting from the formula derived in class, $f(x) = f(a) + \cdots$, replace $x \to x + h, a \to x, \ldots$

Question 45 Solution

a) let $y = 0 \implies x = \pm(1 + \epsilon)$, let $x = 0 \implies y = \pm 1$

b) Solve $y \implies y = f(x) = \pm\sqrt{1 - \left(\frac{x}{1+\epsilon}\right)^2}$

$A(\epsilon) = 2\int_{-1-\epsilon}^{1+\epsilon} \sqrt{1 - \left(\frac{x}{1+\epsilon}\right)^2}dx = 4\int_0^{1+\epsilon} \sqrt{1 - \left(\frac{x}{1+\epsilon}\right)^2}dx$

c)

$A(\epsilon) = 4\int_0^{1+\epsilon} \sqrt{1 - \frac{1}{2}\cdot\left(\frac{x}{1+\epsilon}\right)^2}dx = 4(1+\epsilon)\int_0^{1+\epsilon} \sqrt{1 - \frac{1}{2}\cdot\left(\frac{x}{1+\epsilon}\right)^2}d\frac{x}{1+\epsilon} = 4(1+\epsilon)\int_0^1 \sqrt{1-u^2}du =$ $4(1+\epsilon)\frac{\pi}{4} = (1+\epsilon)\pi$

The first two nonzero terms are $\pi + \pi\epsilon$.

Question 46 Solution

$V(x) = \frac{Gm_1}{|x-x_1|} + \frac{Gm_2}{|x-x_2|}$ for $x \to \infty$ ie., $x > x_1$ and $x > x_2 \implies V(x) = \frac{Gm_1}{x-x_1} + \frac{Gm_2}{x-x_2}$

Using the hint set $y = 1/x$, ie., $x = 1/y$ and expand the potential in powers of y

$V(1/y) = \frac{Gm_1}{1/y-x_1} + \frac{Gm_2}{1/y-x_2} = \frac{Gm_1 y}{1-x_1 y} + \frac{Gm_2 y}{1-x_2 y}$

Using Geometric Series Formula $\frac{1}{1-x_i y} = \sum_{n=0}^{\infty}(x_i y)^n$ where $i = 1, 2$

$V(1/y) = Gm_1 y\sum_{n=0}^{\infty}(x_1 y)^n + Gm_2 y\sum_{n=0}^{\infty}(x_2 y)^n = Gm_1\sum_{n=0}^{\infty}x_1^n y^{n+1} + Gm_2\sum_{n=0}^{\infty}x_2^n y^{n+1} = (Gm_1 +$ $Gm_2)y + (Gm_1 x_1 + Gm_2 x_2)y^2 + (Gm_1 x_2^2 + Gm_2 x_2^2)y^3 + \cdots$

change $y = \frac{1}{x}$ back

$V(x) = (Gm_1 + Gm_2)\frac{1}{x} + (Gm_1 x_1 + Gm_2 x_2)\frac{1}{x^2} + (Gm_1 x_2^2 + Gm_2 x_2^2)\frac{1}{x^3} + \cdots$

Thus $a = (Gm_1 + Gm_2)$, $b = (Gm_1 x_1 + Gm_2 x_2)$, and $c = (Gm_1 x_2^2 + Gm_2 x_2^2)$

Question 47 Solution

Find the quadratic Taylor approximation at $\underline{x = x_0}$, ie., $c_0 + c_1(x - x_0) + c_2(x - x_0)^2$

using the Theorem

$(1+x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 + \cdots$ for $-1 < x < 1$

$V(x) = V_0\left[\left(\frac{x_0}{x}\right)^{12} - 2\left(\frac{x_0}{x}\right)^6\right] = V_0\left[\left(\frac{x}{x_0}\right)^{-12} - 2\left(\frac{x}{x_0}\right)^{-6}\right] = V_0\left[\left(\frac{x-x_0+x_0}{x_0}\right)^{-12} - 2\left(\frac{x-x_0+x_0}{x_0}\right)^{-6}\right] =$ $V_0\left[\left(1 + \frac{x-x_0}{x_0}\right)^{-12} - 2\left(1 + \frac{x-x_0}{x_0}\right)^{-6}\right]$

using the above Theorem

$\left(1 + \frac{x-x_0}{x_0}\right)^{-12} = 1 - 12\frac{x-x_0}{x_0} + \frac{(-12)\cdot(-12-1)}{2}\left(\frac{x-x_0}{x}\right)^2 + \cdots = 1 - \frac{12}{x_0}(x-x_0) + \frac{78}{x_0^2}(x-x_0)^2 + \cdots$

$\left(1 + \frac{x-x_0}{x_0}\right)^{-6} = 1 - 6\frac{x-x_0}{x_0} + \frac{(-6)\cdot(-6-1)}{2}\left(\frac{x-x_0}{x}\right)^2 + \cdots = 1 - \frac{6}{x_0}(x-x_0) + \frac{21}{x_0^2}(x-x_0)^2 + \cdots$

$V(x) = V_0\left[\left(1 - \frac{12}{x_0}(x-x_0) + \frac{78}{x_0^2}(x-x_0)^2 + \cdots\right) - 2\left(1 - \frac{6}{x_0}(x-x_0) + \frac{21}{x_0^2}(x-x_0)^2\right)\right] + \cdots$

$= V_0\left[-1 + \frac{36}{x_0^2}(x-x_0)^2\right] + \cdots = -V_0 + 36\frac{V_0}{x_0^2}(x-x_0)^2 + \cdots$

$T_2(x) = -V_0 + 36\frac{V_0}{x_0^2}(x-x_0)^2$

Question 48 Solution

using the Theorem

$(1+x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + \cdots$ for $-1 < x < 1$

$(1+x^2)^k = 1 + kx^2 + \frac{k(k-1)}{2}x^4 + \frac{k(k-1)(k-2)}{3!}x^6 + \frac{k(k-1)(k-2)(k-3)}{4!}x^8 + \cdots$

$(1+x^2)^{\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}x^4 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^6 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4!}x^8 + \cdots = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \frac{5}{128}x^8 + \cdots$, it is an alternating series.

Using the first order Taylor approximation $T_1(x) = 1$, $|S - T_1| \le a_2 = \frac{1}{2}x^2$ where S denotes the exact value.

$\int_0^1 \sqrt{1+x^2}dx \approx \int_0^1 1dx = x|_0^1 = 1$

The error is bound by $\int \frac{1}{2}x^2 dx = \frac{1}{6}x^3\Big|_0^1 = \frac{1}{6}$

binomial series

Question 49 Solution

a) Show that $\binom{k+1}{n+1} = \binom{k}{n} + \binom{k}{n+1}$

This is true since the left hand side is number of ways of choosing $n+1$ objects from a set of $k+1$ objects (disregarding the order in which the objects are chosen).

The right hand side means that: assume all $k+1$ objects are white, one may randomly pick one object from the $k+1$ objects, coloring it red, then put it back. Now choose $n+1$ objects from these $k+1$ objects, there are two different situations: one situation is that the red one is chosen, the number of ways is $\binom{k}{n}$ (it is equivalent to choosing n from k objects); the other situation is the red one is not chosen, the number of ways is $\binom{k}{n+1}$ (it is equivalent to choosing $n+1$ objects from k objects).

The left hand side equals the right hand side, since it is the same thing, choosing $n+1$ objects from $k+1$ objects.

$\binom{k+1}{n+1} = \frac{(k+1)!}{(n+1)!(k-n)!} = \frac{k!\cdot(k+1)}{(n+1)!(k-n)!} = \frac{k!\cdot(k-n+n+1)}{(n+1)!(k-n)!} == \frac{k!\cdot(k-n)+k!\cdot(n+1)}{(n+1)!(k-n)!} = \frac{k!\cdot(k-n)}{(n+1)!(k-n)!} + \frac{k!\cdot(n+1)}{(n+1)!(k-n)!} = \frac{k!}{(n+1)!(k-n-1)!} + \frac{k!}{n!(k-n)!} = \binom{k}{n+1} + \binom{k}{n}$

b) $\binom{0}{0}$ 1

$\binom{1}{0}\ \binom{1}{1}$ 1 1

$$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \qquad 1\quad 2\quad 1$$

$$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \qquad 1\quad 3\quad 3\quad 1$$

$$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \qquad 1\quad 4\quad 6\quad 4\quad 1$$

$$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \qquad 1\quad 5\quad 10\quad 10\quad 5\quad 1$$

$$\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6} \qquad 1\quad 6\quad 15\quad 20\quad 15\quad 6\quad 1$$

Denote elements in the triangle as $a_{k,n}$, nth element on kth row. Each subsequent row is obtained by adding the two entries diagonally above,

$a_{k+1,n+1} = a_{k,n} + a_{k,n+1}$, ie, $\binom{k+1}{n+1} = \binom{k}{n} + \binom{k}{n+1}$.

c) The next two rows are added in b).

$(a+b)^6 = \binom{6}{0}a^6 + \binom{6}{1}a^5b + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}ab^5 + \binom{6}{6}b^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

complex numbers

Question 50 Solution

a) $1+i \quad (x=1, y=1$ already in Cartesian form$)$

b) $(1+i)^2 = 1 + 2i + i^2 = 1 + 2i - 1 = 2i \quad (x=0, y=2)$

c) $(1+i)^3 = (1+i)^2(1+i) = 2i(1+i) = -2+2i \quad (x=-2, y=2)$

d) $\frac{1}{1+i} = \frac{1\cdot(1-i)}{(1+i)\cdot(1-i)} = \frac{1+i}{1-i^2} = \frac{1+i}{1-(-1)} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i \quad (x=\frac{1}{2}, y=\frac{1}{2})$

e) $\sqrt{1+i} = \left(\sqrt{2}e^{\frac{\pi}{4}i}\right)^{\frac{1}{2}} = 2^{\frac{1}{4}}e^{\frac{\pi}{8}i} = 2^{\frac{1}{4}}\left(\cos\frac{\pi}{8} + i\,\sin\frac{\pi}{8}\right) = 2^{\frac{1}{4}}\cos\frac{\pi}{8} + i\,2^{\frac{1}{4}}\sin\frac{\pi}{8} \quad (x = 2^{\frac{1}{4}}\cos\frac{\pi}{8}, y = 2^{\frac{1}{4}}\sin\frac{\pi}{8})$

Question 51 Solution

a) See 49 c) $(1+i)^6 = 1+6\cdot i+15\cdot i^2+20\cdot i^3+15\cdot i^4+6\cdot i^5+i^6 = 1+6i-15-20i+15+6i-1 = -8i$

b) $1+i = \sqrt{2}e^{\frac{\pi}{4}i}$, since $x=1$, $y=1$, $r=\sqrt{x^2+y^2}=\sqrt{2}$, $\theta = \arctan\frac{y}{x} = \arctan 1 = \frac{\pi}{4}$, $re^{\theta i} = \sqrt{2}e^{\frac{\pi}{4}i}$

$(1+i)^6 = (\sqrt{2}e^{\frac{\pi}{4}i})^6 = 2^{\frac{6}{2}}e^{\frac{6}{4}\pi i} = 2^3e^{\frac{3}{2}\pi i} = 8(\cos\frac{3}{2}\pi + i\,\sin\frac{3}{2}\pi) = -8i$

Question 52 Solution

a) $z^2+2z-2=0 \implies a=1,\ b=2,\ c=-2,\ z_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-2\pm\sqrt{4+8}}{2} = -1\pm\sqrt{3}$ two real roots

b) $z^2+2z+2=0 \implies a=1,\, b=2,\, c=2,\, z_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-2\pm\sqrt{4-8}}{2}=\frac{-2\pm\sqrt{-4}}{2}=\frac{-2\pm2\sqrt{-1}}{2}=$ $-1\pm\sqrt{-1}=-1\pm i$

c) $z^2=1 \implies z_{1,2}=\pm1$

d) $z^3=1 \implies$ three roots: $z_1=1$ and $z_{2,3}=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

e) $z^4=1 \implies z^2=\pm1 \implies$ four roots: $z_{1,2}=\pm1,\, z_{3,4}=\pm i$

f) $e^z=1$ on real axis there is on root $z=0$, but in complex plane there are infinite roots, let $z=x+yi$, $e^z=e^{x+yi}=e^x(\cos y+i\,\sin y)=1 \implies x=0$ and $y=2k\pi$, where k is any integer, roots are $z_k=2k\pi\, i$.

Question 53 Solution

$$(\cos\theta+i\,\sin\theta)^n=\left(e^{\theta i}\right)^n=e^{n\theta i}=\cos n\theta+i\,\sin n\theta$$

Question 54 Solution

$\int e^{ax}\cos bx\,dx=\int e^{ax}\frac{1}{b}\,d\sin bx=\frac{1}{b}e^{ax}\cdot\sin bx-\int\frac{1}{b}\sin bx\,de^{ax}=\frac{1}{b}e^{ax}\cdot\sin bx-\int\frac{a}{b}e^{ax}\sin bx\,dx=$ $\frac{1}{b}e^{ax}\cdot\sin bx-\int\frac{a}{b}e^{ax}\left(-\frac{1}{b}\right)\,d\cos bx=\frac{1}{b}e^{ax}\cdot\sin bx+\int\frac{a}{b^2}e^{ax}\,d\cos bx=\frac{1}{b}e^{ax}\cdot\sin bx+\frac{a}{b^2}e^{ax}\cdot\cos bx-$ $\frac{a}{b^2}\int\cos bx\,de^{ax}=\frac{1}{b}e^{ax}\cdot\sin bx+\frac{a}{b^2}e^{ax}\cdot\cos bx-\frac{a^2}{b^2}\int e^{ax}\cos bx\,dx \implies \underline{\int e^{ax}\cos bx\,dx}=\frac{1}{b}e^{ax}\cdot$ $\sin bx+\frac{a}{b^2}e^{ax}\cdot\cos bx-\frac{a^2}{b^2}\underline{\int e^{ax}\cos bx\,dx} \implies \left(1+\frac{a^2}{b^2}\right)\int e^{ax}\cos bx\,dx=\frac{1}{b}e^{ax}\cdot\sin bx+\frac{a}{b^2}e^{ax}\cdot\cos bx$

$\implies \int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}\left(b\cdot\sin bx+a\cdot\cos bx\right)+C$

$\int e^{ax}\sin bx\,dx=\int e^{ax}\left(-\frac{1}{b}\right)\,d\cos bx=-\frac{1}{b}e^{ax}\cdot\cos bx+\int\frac{1}{b}\cos bx\,de^{ax}=-\frac{1}{b}e^{ax}\cdot\cos bx+\int\frac{a}{b}e^{ax}\cos bx\,dx=$ $-\frac{1}{b}e^{ax}\cdot\cos bx+\int\frac{a}{b}e^{ax}\cdot\frac{1}{b}\,d\sin bx=-\frac{1}{b}e^{ax}\cdot\cos bx+\int\frac{a}{b^2}e^{ax}\,d\sin bx=-\frac{1}{b}e^{ax}\cdot\cos bx+\frac{a}{b^2}e^{ax}\cdot\sin bx-$ $\frac{a}{b^2}\int\sin bx\,de^{ax}=-\frac{1}{b}e^{ax}\cdot\cos bx+\frac{a}{b^2}e^{ax}\cdot\sin bx-\frac{a^2}{b^2}\int e^{ax}\sin bx\,dx \implies \underline{\int e^{ax}\sin bx\,dx}=-\frac{1}{b}e^{ax}\cdot$ $\cos bx+\frac{a}{b^2}e^{ax}\cdot\sin bx-\frac{a^2}{b^2}\underline{\int e^{ax}\sin bx\,dx} \implies \left(1+\frac{a^2}{b^2}\right)\int e^{ax}\sin bx\,dx=-\frac{1}{b}e^{ax}\cdot\cos bx+\frac{a}{b^2}e^{ax}\cdot\sin bx$

$\implies \int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}\left(a\cdot\sin bx-b\cdot\sin bx\right)+C$

b) $e^{(a+ib)x}=e^{ax+ibx}=e^{ax}\cdot e^{ibx}=e^{ax}\left(\cos bx+i\,\sin bx\right)$

$\int e^{(a+ib)x}dx=\frac{1}{a+ib}e^{(a+ib)x}=\frac{1\cdot(a-ib)}{(a+ib)\cdot(a-ib)}e^{(a+ib)x}=\frac{a-ib}{a^2+b^2}e^{(a+ib)x}$

since $\int e^{(a+ib)x}dx=\int\left(e^{ax}\cos bx+i\,e^{ax}\sin bx\right)dx=\int e^{ax}\cos bx\,dx+i\int e^{ax}\sin bx\,dx=\frac{a-ib}{a^2+b^2}e^{(a+ib)x}=$ $\frac{a-ib}{a^2+b^2}\left(e^{ax}\cos bx+i\,e^{ax}\sin bx\right)=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+i\left[\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\sin bx)\right]$

Thus $\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)$ and $\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\sin bx)$

Question 55 Solution

a) since $e^{ix}=\cos x+i\,\sin x$ and $e^{-ix}=\cos x-i\,\sin x \implies e^{ix}+e^{-ix}=2\cos x \implies \cos x=$

$\frac{e^{ix}+e^{-ix}}{2}$

b) $e^{ix} - e^{-ix} = 2i\sin x \implies \sin x = \frac{e^{ix}-e^{-ix}}{2i}$

c) $\frac{d}{dx}\cos x = \frac{d}{dx}\left(\frac{e^{ix}+e^{-ix}}{2}\right) = \frac{ie^{ix}-ie^{-ix}}{2} = \frac{\left(ie^{ix}-ie^{-ix}\right)i}{2i} = \frac{-e^{ix}+e^{-ix}}{2i} = -\sin x$

d) $\frac{d}{dx}\sin x = \frac{d}{dx}\left(\frac{e^{ix}-e^{-ix}}{2i}\right) = \frac{ie^{ix}+ie^{-ix}}{2i} = \frac{e^{ix}+e^{-ix}}{2} = \cos x$

e) $1 = e^{ix} \cdot e^{-ix}(\cos x + i\sin x) \cdot (\cos x - i\sin x) = \cos^2 x - i^2\sin^2 x = \cos^2 x + \sin^2 x$

f) $e^{2xi} = \cos 2x + i\ \sin 2x$

$e^{2xi} = e^{xi} \cdot e^{xi} = (\cos x + i\sin x)(\cos x + i\sin x) = \cos^2 x + i \cdot 2\cos x\sin x + i^2\sin^2 x = \cos^2 x - \sin^2 x + i \cdot 2\cos x\sin x$

Thus $\cos 2x = \cos^2 x - \sin^2 x$ (g) and $\sin 2x = 2\sin x\cos x$