Math 156 Applied Honors Calculus II Fall 2009

hw8 , due: Tuesday, November 10

section 9.5 (probability) page 617 / 9 , 13 , 14
note: you may use Maple or a calculator to evaluate the integral in problem 13.

section 10.1 (differential equations) page 627 / 3 , 4 , 9

section 10.4 (exponential growth and decay) page 656 / 3 , 7 , 20

1. Let X be a random variable. Show that $\sigma(X)^2 = \mu(X^2) - \mu(X)^2$. (Note: if X is a random variable with pdf $f(x)$, then X^2 is also a random variable with pdf $f(x)$, i.e. you may assume $\mu(X) = \int_{-\infty}^{\infty} x f(x) dx$ and $\mu(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$.)

2. Consider a given function $f(x)$ and a point $x = a$. In this exercise, x is a variable and a is a constant. We will derive a linear approximation to $f(x)$ near the point $x = a$.

a) Show that $f(x) = f(a) + f'(a)(x-a) + \int_a^x (x-t) f''(t)\, dt$. (hint: start with the integral term and integrate by parts with $u = x - t$, $dv = f''(t)\, dt$.)

b) Define the function $T_1(x)$ as below.

$$T_1(x) = f(a) + f'(a)(x-a)$$

Note that $T_1(x)$ is a linear function of x; it is called the <u>Taylor polynomial of degree 1</u> for $f(x)$ at $x = a$. Show that $T_1(x)$ and $f(x)$ have the same function value and 1st derivative value at $x = a$. This implies that $T_1(x)$ is tangent to the graph of $f(x)$ at $x = a$ and hence we view $T_1(x)$ as a <u>linear approximation</u> to $f(x)$ near the point $x = a$.

c) Note that part (a) says,

$$f(x) \;=\; T_1(x) + \int_a^x (x-t) f''(t)\, dt.$$

The <u>error</u> is the difference between the given function $f(x)$ and the linear approximation $T_1(x)$. Show that the error satisfies the bound $|f(x) - T_1(x)| \le \frac{1}{2} M_2 |x-a|^2$, where $M_2 = \max |f''(t)|$. This implies that if x is close to a, then $T_1(x)$ is a good approximation to $f(x)$.

d) In each case below find $T_1(x)$ and sketch $f(x)$, $T_1(x)$ on the same graph.

(i) $f(x) = e^x,\ a = 0$ (ii) $f(x) = \sin x,\ a = 0$ (iii) $f(x) = \sin x,\ a = \frac{\pi}{4}$

3. a) Show that $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$. (hint: set $\sinh^{-1} x = y$, so that $x = \sinh y$, then set $u = e^y$ and solve for u in terms of x, then substitute back to obtain y in terms of x)

b) The antiderivative $\int \frac{dx}{\sqrt{x^2+1}} = \ln(x + \sqrt{x^2+1}\,)$ was derived in class using the trig substitution $x = \tan\theta$ (this came up in connection with the arclength of a parabola). Rederive the antiderivative using the substitution $x = \sinh y$.

4. Consider Newton's law of cooling/heating, $y' = k(y - T)$. In class we derived the solution $y(t) = T + (y_0 - T)e^{kt}$ by separation of variables. Check that the given expression for $y(t)$ does in fact satisfy the differential equation.

Announcement

The Math Department is interested in attracting more majors. If you're already committed to another major, then please consider doing a math minor - it requires only 101 math classes (in base 2) beyond Math 156. www.math.lsa.umich.edu/undergrad/minor.shtml