Addendum to lecture 4

We want to prove part 1 of the Fundamental Theorem of Calculus (FTC):

Theorem. *If f is continuous on $[a,b]$, then the function $F(x) = \int_a^x f(t)dt$ satisfies $F'(x) = f(x)$ for x in $[a,b]$.*

Proof. We compute

$$\begin{aligned} F(x+h) &= \int_a^{x+h} f(t)dt \qquad \text{(by definition)} \\ &= \int_a^x f(t)dt + \int_x^{x+h} f(t)dt \qquad \text{(by property 3 of integrals)} \\ &= F(x) + \int_x^{x+h} f(t)dt \qquad \text{(by definition).} \end{aligned}$$

Thus

$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)dt.$$

In class we argued a bit vaguely that $\int_x^{x+h} f(t)dt \approx hf(x)$, so indeed $F'(x) \approx f(x)$. This can be made rigorous in a few ways. One is done in the book (pp. 342–343), by means of the Extreme Value Theorem and the Squeeze Theorem from Calculus I. Here is the argument I sketched in class: since f is continuous at x, for any $\epsilon > 0$ there exists $\delta > 0$ such that $f(x) - \epsilon < f(t) < f(x) + \epsilon$ whenever $x - \delta < t < x + \delta$. Choosing h such that $0 < h < \delta$, and using property 4 of integrals, it follows that

$$\int_x^{x+h} (f(x) - \epsilon) \leq \int_x^{x+h} f(t)dt \leq \int_x^{x+h} (f(x)+\epsilon)dt,$$

or in other words

$$h(f(x)-\epsilon) \leq \int_x^{x+h} f(t)dt \leq h(f(x)+\epsilon).$$

Thus

$$f(x) - \epsilon \leq \frac{1}{h}\int_x^{x+h} f(t)dt \leq f(x) + \epsilon.$$

Since this holds for every h with $0 < h < \delta$, the definition of limits implies that

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)dt = f(x),$$

so indeed $F'(x) = f(x)$ as desired. □

Let me know if you have any questions!