Compound interest

Suppose you invest x dollars (your *principal* at an annual interest rate of r (where $0 < r < 1$).

If interest is compounded annually then:

- after one year, you'll have $x + rx = (1 + r)x$ dollars;
- after two years, you'll have $(1 + r)^2x$ dollars;
- after t years, you'll have $(1 + r)^tx$ dollars.

If interest is compounded semiannually (two times a year), then:

- after six months, you'll have $x + \frac{r}{2}x = (1 + \frac{r}{2})x$ dollars;
- after one year, you'll have $(1 + \frac{r}{2})^2x$ dollars;
- after t years, you'll have $(1 + \frac{r}{2})^tx$ dollars.

Note that $(1 + \frac{r}{2})^2 = 1 + r + \frac{r^2}{4}$, so after one year of semiannual compounding you'll have $\frac{r^2}{4}x$ more dollars than after one year of annual compounding (because the interest earned in the first six months then earns interest itself in the second six months). Thus there is an advantage to compounding as often as possible.

If interest is compounded daily (in a non-leap year), then

- after one day, you'll have $x + \frac{r}{365}x = (1 + \frac{r}{365})x$ dollars;
- after two days, you'll have $(1 + \frac{r}{365})^2x$ dollars;
- after one year, you'll have $(1 + \frac{r}{365})^{365}x$ dollars;
- after t years, you'll have $(1 + \frac{r}{365})^{365t}x$ dollars.

If you continue in this manner, compounding every minute, or every second, or every millisecond, then the limit is *continuously compounded interest*, in which after t years you'll have

$$\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{nt} x = x \left(\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{n}\right)^{t}$$

dollars. On the homework you'll show that

$$\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{n} = e^r,$$

so the value of your investment after t years is $x(e^r)^t = xe^{rt}$.

In other words, if we write $y(t)$ for the amount of money you have after t years of continuous compounding at annual interest rate r, then $y(t) = y(0)e^{rt}$, which is a solution to the differential equation $y' = ry$ (which makes intuitive sense if you think about it).

For example, if you invest \$1000 for ten years at a 10% annual interest rate, then $x = 1000$, $r = 0.10$, and $t = 10$, so:

- with annual compounding, you'll wind up with $(1 + r)^tx = (1.1)^{10} \cdot 1000 = \$2593.74$;
- with continuous compounding, you'll wind up with $e^{rt}x = e^{0.1\cdot 10} \cdot 1000 = \$2718.28$.

In this example, you can check that $e^{0.1} \approx 1.1052$, which means that continous compounding at a 10% annual interest rate is equivalent to annual compounding at a 10.52% annual interest rate (this is the *equivalent annual interest rate*).

Let me know if you have any questions!