

# Math 116 – Team Homework #1, Winter 2023

## SOLUTIONS

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### Some guidelines for your first assignment

- You must *read* and *attempt* the problems *before* meeting with your team. Even if you aren't able to obtain all the answers, being prepared during the team meetings helps your group work more efficiently during the meeting.
  - Don't be discouraged if you cannot solve most of the problems on your own — this is perfectly normal. This is part of why you are being assigned to work on these assignments as a group; make sure to discuss your questions and ideas with your teammates.
  - If your team is having trouble with a particular problem, try utilizing the Math Lab (our math tutoring center - see more details here: <https://lsa.umich.edu/math/undergraduates/course-resources/math-lab.html>) with your teammates to get help.
  - Make sure *everyone* is involved and no-one feels excluded during the meetings. If you notice someone is shy, actively encourage them to contribute to the group!
  - Ask your teammates to explain their reasoning behind their answers if you don't understand it. Remember that all members of the team are responsible for this assignment, and *everyone* should be on board with what the team turns in.
  - Write up your final solutions neatly, and make sure your explanations are clear and complete.
  - Consult pages 12-14 of the Student Guide on the course website for more details regarding best practices and team homework roles.
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1. In her spare time, Zara sketches designs for what she believes to be efficient traffic interchanges<sup>1</sup>. Zara's father runs computer simulations on her designs to model their efficiency.

One day, Zara tries to count the number of vehicles passing through the interchange in front of their house. Zara quickly realized that it is going to be a difficult task to sit and count vehicles, no matter how slow, for every minute of the day. She decided that she would only count the number of vehicles passing through the interchange over the course of the first 15 minutes (of each hour).

The data that Zara collected between 6am and 12:15pm is presented in the table below, where  $N(t)$  is the number of vehicles passing through the interchange in the first 15 minutes of the  $t$ -th hour after 6:00 am, e.g.  $N(0)$  is the number of vehicles passing through the interchange between 6:00 am and 6:15 am.

$t$	0	1	2	3	4	5	6
$N(t)$	42	93	120	132	117	94	42

- (a) Zara uses left and right Riemann sums, each with six subdivisions, to come up with estimates for the **total** number of vehicles passing through the interchange from 6:00 am until noon. Find values for the estimates, LEFT(6) and RIGHT(6), that Zara computes. (*Be careful with the units!*) What are you assuming about the number of cars that pass through the interchange each hour in the LEFT(6) estimate? What are you assuming about the number of cars that pass through the interchange each hour in the RIGHT(6) estimate?

**Solution:** Remember that  $N(t)$  measure the number of vehicles passing through the interchange in the first 15 minutes of each hour, and so if we want to estimate the number of vehicles passing through the interchange from  $t$  hours after 6am till  $t + 1$  hours after 6am, we can use either  $4N(t)$  (for a LEFT estimate) or  $4N(t + 1)$  (for a RIGHT estimate). We have:

$$\begin{aligned} \text{LEFT}(6) &= 4N(0) + 4N(1) + 4N(2) + 4N(3) + 4N(4) + 4N(5) \\ &= 4(42) + 4(93) + 4(120) + 4(132) + 4(117) + 4(94) \\ &= 168 + 372 + 480 + 528 + 468 + 376 \\ &= 2392. \end{aligned}$$

and similarly,

$$\begin{aligned} \text{RIGHT}(6) &= 4N(1) + 4N(2) + 4N(3) + 4N(4) + 4N(5) + 4N(6) \\ &= 4(93) + 4(120) + 4(132) + 4(117) + 4(94) + 4(42) \\ &= 372 + 480 + 528 + 468 + 376 + 168 \\ &= 2392. \end{aligned}$$

For the LEFT estimate, we have assumed that the average rate that vehicles pass through intersection in the first 15 minutes of each hour is the same as the average rate that vehicles pass through the intersection for the remainder of that hour.

For the RIGHT estimate, we have assumed that the average rate that vehicles pass through the intersection in the first 15 minutes of each hour is the same as the average rate that vehicles passed through the intersection in the preceding hour.

- (b) What do your answers in part (a) say, if anything, about the actual value of the total number of vehicles passing through the interchange from 6:00 am until noon?

**Solution:** We saw that both estimates gave a value of 2392 cars. While 2392 is an estimate to the number of vehicles which passed through the intersection, we can't say how accurate an

<sup>1</sup>YouTube: Traffic flow measured on 30 different 4-way junctions

estimate it is, or whether it is an over- or under-estimate to the true number of vehicles which passed through the intersection between 6:00am and noon.

If we knew that the rate that vehicles passed through the intersection was monotonic (i.e. always increasing or always decreasing), then we could say for sure that one of LEFT(6) and RIGHT(6) would be an overestimate and the other would be an underestimate. In this problem though,  $N(t)$  sometimes increases and sometimes decreases, and so it is possible that both estimates are underestimates, or both estimates are overestimates.

Zara's father attempts to fit a function through Zara's data, and notices that  $N(t)$  can be modeled extremely well by  $n(t) = 42 - 10t(t - 6)$ , for  $0 \leq t \leq 6$ .

- (c) Using only  $n(t)$ , what is the best estimate you can find for the total number of vehicles passing through the interchange from 6:00 am until noon? (*Again, be very careful with the units. What are the units of  $n(t)$ ?*) How does this compare with your answer in part (a)? In particular, presuming the estimate using  $n(t)$  is no more than 50 vehicles away from the actual value, were LEFT(6) and RIGHT(6) over- or under-estimates of the actual value?

**Solution:** Note that the units of  $n(t)$  should have the same units as  $N(t)$ , which was measured in vehicles per 15 minutes. Therefore  $4n(t)$  is measured in vehicles per hour, and we can find an estimate to the total number of cars passing through the intersection from 6:00am to noon using:

$$\begin{aligned}\int_0^6 4n(t)dt &= 4 \int_0^6 42 - 10t(t - 6) dt \\ &= 4 \int_0^6 (42 - 10t^2 + 60t) dt \\ &= 4 \left( 42t - \frac{10}{3}t^3 + 30t^2 \right) \Big|_0^6 \\ &= 4 \left( 42(6) - \frac{10}{3}(6)^3 + 30(6)^2 \right) \\ &= 4(252 - 720 + 1080) \\ &= 4(612) \\ &= 2448.\end{aligned}$$

Presuming that this estimate is accurate to within 50 vehicles, we see that somewhere between 2398 and 2498 vehicles passed through the intersection between 6:00am and noon, meaning that both LEFT(6) and RIGHT(6) gave us underestimates.

- (d) Traffic flow is defined as the number of vehicles passing through the interchange per hour, and is considered a measurement for the efficiency of an interchange. Use  $n(t)$  to estimate the *average* traffic flow from 6:00 am to noon on the interchange, and provide Zara with the number (given by  $n(t)$ ) to beat with her design!

**Solution:** If the total number of vehicles passing through the intersection in the 6 hour period was 2448 (as we calculated in part (c)) then the average number of vehicles which passed through the intersection was  $2448/6=408$  vehicles per hour.

2. Ailim is a muralist who creates her murals by assembling irregularly-shaped tiles<sup>2</sup>. Anna, an art enthusiast and a tile collector, notices one of Ailim’s works at the Eden Gallery. Enamored by its perfection, Anna wishes to recreate the masterpiece at home.

Anna designs each tile of Ailim’s mural on her computer, and recreates it at her friend’s workshop. Anna uploads several pictures of the mural to her computer, and she has a program that attempts to outline each tile using the photos. The program works remarkably well for most of the mural, but the software encounters difficulties in a small section of the mural with minuscule tiles. The software models top edge of each minuscule tile by a part of the graph of the function  $E(x) = \sqrt{1 - x^2}$ . The sides and bottom edges of the minuscule tiles appear to be essentially straight.

Anna uses this information, along with area results from her sensitive planimeter (a device used to measure the area of two dimensional shapes), to accurately outline these tiles. According to the software model, each minuscule tile has area<sup>3</sup>

$$A(t) = \int_{-2t}^{1-t} E(x) dx = \int_{-2t}^{1-t} \sqrt{1 - x^2} dx,$$

for some number  $t$ , where  $0 \leq t \leq \frac{1}{2}$ . The units of  $t$  and  $x$  are millimeters, and the units of  $A(t)$  are square millimeters. Note that, for any fixed  $t$ , the following outline for the (minuscule) tile is implicitly defined by the expression for  $A(t)$ .

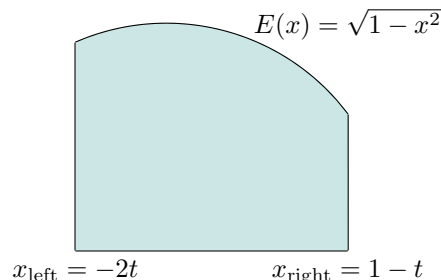


Figure 1: Outline of a (minuscule) tile with area given by  $A(t)$

**Use computational software to find the value of any integrals you need for either part of the problem.**

- (a) Anna finds that her planimeter outputs the area of one of the tiles to be just slightly over the maximum area allowed by  $A$ . Instead of using the area given by the planimeter, she decides to use the area given by the maximum of  $A(t)$ . For what value of  $t$  does  $A(t)$  attain its maximum? What are the values of  $x_{\text{left}}$  and  $x_{\text{right}}$ , as defined in Figure 1, when  $A(t)$  is at a maximum?

**Solution:** To find where  $A(t)$  has its maximum, we first look for the critical points of  $A(t)$ . Applying the Chain Rule for differentiation and the Second Fundamental Theorem of Calculus, we see that

$$A'(t) = -\sqrt{1 - (1 - t)^2} + 2\sqrt{1 - 4t^2}.$$

On the interval  $0 \leq t \leq \frac{1}{2}$ , both  $1 - (1 - t)^2$  and  $1 - 4t^2$  are nonnegative, and so  $A'(t)$  is always defined on its domain. To find critical points, we must then look for where  $A'(t) = 0$ , i.e.

$$-\sqrt{1 - (1 - t)^2} + 2\sqrt{1 - 4t^2} = 0.$$

So,

$$2\sqrt{1 - 4t^2} = \sqrt{1 - (1 - t)^2}$$

<sup>2</sup>Quanta Magazine: ‘Nasty’ Geometry Breaks Decades-Old Tiling Conjecture

<sup>3</sup>Anna notes that the bounds for  $A$  ensure that she never attempts to integrate the function  $E$  outside its domain. Furthermore, she thinks it allows her just enough flexibility to outline the tiles at hand.

and hence, squaring both sides,

$$4(1 - 4t^2) = 1 - (1 - t)^2.$$

Rearranging, we see that

$$15t^2 + 2t - 4 = 0,$$

and by the quadratic formula, we see

$$t = \frac{-2 \pm \sqrt{244}}{30} = \frac{-1 \pm \sqrt{61}}{15}.$$

Thus,  $t \approx -0.587$  or  $t \approx 0.454$ . The first is not in the correct domain, so the only critical point of  $A(t)$  is

$$t = \frac{-1 + \sqrt{61}}{15} \approx 0.454.$$

It is easy to verify that  $A'(0) = 2$  and  $A'(\frac{1}{2}) = -\sqrt{\frac{3}{4}}$ , and so by the First Derivative Test, the critical point we found is indeed a local maximum, and since it is the only critical point, it must be a global maximum for  $A(t)$ .

For this value of  $t$ , we have

$$x_{\text{left}} = -2 \left( \frac{-1 + \sqrt{61}}{15} \right) = \frac{2 - 2\sqrt{61}}{15} \approx -0.908$$

and

$$x_{\text{right}} = 1 - \left( \frac{-1 + \sqrt{61}}{15} \right) = \frac{16 - \sqrt{61}}{15} \approx 0.546.$$

While the question does not specify that we should calculate this, we may see using an integral calculator that

$$A \left( \frac{-1 + \sqrt{61}}{15} \right) \approx \int_{-0.908}^{0.546} \sqrt{1 - x^2} dx \approx 1.277,$$

and so the maximum area of a tile is approximately  $1.277\text{mm}^2$ .

An alternative to the First Derivative Test here would be to use the Extreme Value Theorem – the maximum value of  $A(t)$  must be located either at a critical point, or at one of the endpoints of its domain. In part (b) of this problem, we calculate the values of  $A(0)$  and  $A(\frac{1}{2})$ , and we can see that both are smaller than  $1.277$ , meaning that we have indeed found the global maximum for the area of a tile at the critical point  $t = \frac{-1 + \sqrt{61}}{15}$ .

- (b) Anna's friend, Brian, owns a workshop and specializes in cutting tiles. Brian's machinery can accurately cut tiles that have area larger than  $\frac{3}{4} \text{mm}^2$ . If Anna presents Brian with a tile outline output from the software model, is it possible that Brian won't be able to cut it?

**Solution:** In part (a), we found the only critical point of  $A(t)$ , and saw that it was a global maximum. The global minimum must then occur at one of the endpoints of the interval  $0 \leq t \leq \frac{1}{2}$ . Using an integral calculator, we see:

$$A(0) = \int_0^1 \sqrt{1 - x^2} dx \approx 0.785.$$

(Note that when  $t = 0$ , the tile is a quarter-circle of radius 1, and so the exact area is  $\frac{\pi}{4} \text{mm}^2$ ).

We can also see:

$$A\left(\frac{1}{2}\right) = \int_{-1}^{1/2} \sqrt{1-x^2} dx \approx 1.2637.$$

(Alternatively, you may have observed that when  $t = \frac{1}{2}$ , the tile includes, but is larger than, a quarter-circle of radius 1, and so must have an area bigger than when  $t = 0$ .)

In conclusion, the smallest tile occurs when  $t = 0$ , and in that case, the area is approximately  $0.785 \text{ mm}^2$ . As this is larger than  $\frac{3}{4} \text{ mm}^2$ , Brian's machinery should always be able to accurately cut the tiles.

**Fun bonus question.** Below is an actual chat conversation between a human (P) and ChatGPT (green logo), a non-human chatbot. **Note that in this conversation, ChatGPT says some things that are true, and other things that are not true.**

- Why do you think (P) says ChatGPT's first answer doesn't make sense? (i.e. What do you think is crossed out?)
- Do you think the second answer is better than the first or worse than the first?
- Can you find all the false statements that ChatGPT says in the conversation?

P What is the integral of  $e^{(x^2)}$ ?

The integral of the function  $e^{(x^2)}$  cannot be expressed in terms of elementary functions. This means that there is no simple formula involving only the basic arithmetic operations (addition, subtraction, multiplication, division), powers, and roots that can be used to evaluate the integral. In general, the integral of a function of the form  $e^{(g(x))}$  can be expressed in terms of the error function erf, which is defined as follows:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Using this definition, the integral of  $e^{(x^2)}$  can be written as:

$$\int e^{(x^2)} dx = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt + C$$

where C is an arbitrary constant of integration. This formula can be used to approximate the value of the integral numerically, but it cannot be evaluated exactly in closed form using only elementary functions.

P That doesn't really make sense because [redacted]  
[redacted] Can you check that the formula is correct?

I'm sorry if my previous response was unclear. The formula that I provided for the integral of  $e^{(x^2)}$  is not an exact formula, but rather an approximation that can be used to evaluate the integral numerically. This means that it will not necessarily give the exact value of the integral for all values of x, but it can be used to compute a numerical approximation to the integral with a specified degree of accuracy.

To compute the exact value of the integral of  $e^{(x^2)}$ , we can use the fact that the derivative of  $e^{(x^2)}$  is given by  $2x * e^{(x^2)}$ . This means that if we can find a function  $f(x)$  such that  $f'(x) = e^{(x^2)}$ , then the function  $f(x)$  will be an antiderivative of  $e^{(x^2)}$ .

One way to find such a function is to use the fact that the derivative of the function  $f(x) = e^{(x^2)}$  is  $2x * e^{(x^2)}$ , which is equal to  $e^{(x^2)}$  up to a constant multiple. This means that the function  $g(x) = e^{(x^2)} / 2$  is also an antiderivative of  $e^{(x^2)}$ . We can then use the formula for the indefinite integral to write:

$$\int e^{(x^2)} dx = \frac{1}{2} \int e^{(x^2)} dx + C = \frac{1}{2} * g(x) + C = \frac{1}{2} * (e^{(x^2)} / 2) + C = \frac{1}{4} * e^{(x^2)} + C$$

where C is an arbitrary constant of integration. This is the exact formula for the indefinite integral of  $e^{(x^2)}$ , and it can be used to evaluate the integral exactly for any given value of x. Note that this formula is different from the approximation that I provided in my previous response, which is only valid for computing numerical approximations to the integral. I apologize if my previous response was unclear or misleading.