Math 116 – Team Homework #2, Winter 2023

SOLUTIONS

Some guidelines for your assignment

- You must *read* and *attempt* the problems *before* meeting with your team. Even if you aren't able to obtain all the answers, being prepared during the team meetings helps your group work more efficiently during the meeting.
- Don't be discouraged if you cannot solve most of the problems on your own — this is perfectly normal. This is part of why you are being assigned to work on these assignments as a group; make sure to discuss your questions and ideas with your teammates.
- If your team is having trouble with a particular problem, try utilizing the Math Lab (our math tutoring center - see more details here: `https://lsa.umich.edu/math/undergraduates/course-resources/math-lab.html`) with your teammates to get help.
- Make sure *everyone* is involved and no-one feels excluded during the meetings. If you notice someone is shy, actively encourage them to contribute to the group!
- Ask your teammates to explain their reasoning behind their answers if you don't understand it. Remember that all members of the team are responsible for this assignment, and *everyone* should be on board with what the team turns in.
- Write up your final solutions neatly, and make sure your explanations are clear and complete.
- Consult pages 12-14 of the Student Guide on the course website for more details regarding best practices and team homework roles.

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1. Clayden and Keisha are university students who work at a pawn shop, part time. Each month, the owner of the shop, Dalon, allows each of them to take one item of moderate value from the shop as part of their compensation. This month, Clayden notices a small portable safe with a piece of paper taped over it. The paper has three integrals on it, and it also says that the answers to the three integrals (entered in order) form the passcode to the safe. Clayden is taking Calculus this semester, so he tells Dalon that he is choosing the safe for his free item for the month. The clues on the safe are:

   - **Clue 1:** The value of the definite integral

     $$\int_0^1 \frac{(\ln(h(x)))^2}{h(x)}\, h'(x)\, dx,$$

     where $h(0) = e^3$, and $h(1) = e^6$.
   - **Clue 2:** The value of the expression

     $$2e^{4\pi} - 65 \int_3^5 e^{4m(x)} \cos\left(\frac{m(x)}{2}\right) m'(x)\, dx,$$

     where $m(3) = 0$, and $m(5) = \pi$.
   - **Clue 3:** The smallest integer larger than the value of the definite integral

     $$\int_{\arctan(2)}^{\arctan(8)} \frac{500}{(\sin^2 x)(\tan x + 2)}\, dx.$$

   However, before letting Clayden take the safe, Dalon wants first to ensure that the safe does not contain any items of excessive value, so Dalon asks Clayden to solve the clues to help him access and assess the safe's contents. Clayden is able to comfortably and correctly solve the first two clues.

   (Note: Throughout this problem, compute all integrals by hand, but feel free to use a computational device to get decimal approximations for values of functions.)

   (a) What is the value he obtains for the definite integral in Clue 1?

   Solution: We can compute this integral by applying u-substitution twice. First, let $u = h(x)$, giving us $du = h'(x)\, dx$. Remembering to change our bounds, $x = 0 \to u = h(0) = e^3$, and $x = 1 \to u = h(1) = e^6$, we get

   $$\int_0^1 \frac{(\ln(h(x)))^2}{h(x)} h'(x)dx = \int_{e^3}^{e^6} \frac{(\ln(u))^2}{u} du.$$

   Now let $v = \ln u$ and $dv = \frac{1}{u}\, du$. Again, remembering to change our bounds, $u = e^3 \to v = \ln(e^3) = 3$, and $u = e^6 \to v = \ln(e^6) = 6$, we get

   $$\int_3^6 v^2 dv = \frac{v^3}{3}\Big|_3^6 = \frac{6^3}{3} - \frac{3^3}{3} = \frac{216-27}{3} = \frac{189}{3} = \boxed{\mathbf{63}}.$$

   Note: Alternatively, the single substitution, $w = \ln(h(x))$ can be applied to the original integral to obtain the same result.

   (b) What is the value he obtains for the expression in Clue 2?

   Solution:
   We can compute this integral by applying w-substitution and then integration by parts. First, let $w = m(x)/2$ and $dw = (1/2)m'(x)\, dx$. Remembering to change our bounds, we get

   $$2e^{4\pi} - 65 \int_3^5 e^{4m(x)} \cos(m(x)/2) m'(x)\, dx = 2e^{4\pi} - 130 \int_0^{\pi/2} e^{8w} \cos(w)\, dw.$$

We will now apply integration by parts to compute $\int_0^{\pi/2} e^{8w}\cos(w)\,dw$. Let

$$u = \cos(w) \qquad v' = e^{8w}$$
$$u' = -\sin(w) \qquad v = (1/8)e^{8w}$$

Then

$$\int_0^{\pi/2} e^{8w}\cos(w)\,dw = \frac{1}{8}\cos(w)e^{8w}\Big|_0^{\pi/2} - \int_0^{\pi/2} -\frac{1}{8}\sin(w)e^{8w}\,dw = -\frac{1}{8} + \int_0^{\pi/2}\frac{1}{8}\sin(w)e^{8w}\,dw.$$

Using integration by parts again, choosing $u_2 = \frac{1}{8}\sin(w)$ so $u_2' = \frac{1}{8}\cos(w)$ and $v_2' = e^{8w}$ with $v_2 = \frac{1}{8}e^{8w}$, we continue

$$\begin{aligned}\int_0^{\pi/2} e^{8w}\cos(w)\,dw &= -\frac{1}{8} + \left(\frac{1}{64}\sin(w)e^{8w}\Big|_0^{\pi/2}\right) - \int_0^{\pi/2}\frac{1}{64}\cos(w)e^{8w}\,dw \\ &= -\frac{1}{8} + \frac{1}{64}e^{4\pi} - \int_0^{\pi/2}\frac{1}{64}\cos(w)e^{8w}\,dw \\ &= -\frac{1}{8} + \frac{1}{64}e^{4\pi} - \frac{1}{64}\int_0^{\pi/2}\cos(w)e^{8w}\,dw.\end{aligned}$$

This gives $\int_0^{\pi/2} e^{8w}\cos(w)\,dw = -\frac{1}{8} + \frac{1}{64}e^{4\pi} - \frac{1}{64}\int_0^{\pi/2}\cos(w)e^{8w}\,dw.$

So $\frac{65}{64}\int_0^{\pi/2} e^{8w}\cos(w)\,dw = -\frac{1}{8} + \frac{1}{64}e^{4\pi}$ and therefore,

$$\int_0^{\pi/2} e^{8w}\cos(w)\,dw = \frac{64}{65}\left(-\frac{1}{8} + \frac{1}{64}e^{4\pi}\right) = \frac{1}{65}\left(-8 + e^{4\pi}\right).$$

So we can conclude that the expression for Clue 2 is equal to

$$2e^{4\pi} - 130\int_0^{\pi/2} e^{8w}\cos(w)\,dw = 2e^{4\pi} - 130\left(\frac{1}{65}\right)\left(-8 + e^{4\pi}\right) = 2e^{4\pi} + 16 - 2e^{4\pi} = \boxed{\mathbf{16}}.$$

Before Clayden is able to solve the third clue, his shift at the pawn shop ends, and he is unable to stay for longer. As Clayden is leaving, Keisha arrives for her shift at the pawn shop. Before leaving, Clayden reminds both Dalon and Keisha that he has dibs on the safe.

Dalon tells Keisha that Clayden has already solved the first two clues, and he would like it if Keisha could solve the third for him.

For the definite integral in Clue 3, Keisha first uses trigonometric identities, then performs an appropriate substitution along with a change of bounds, to obtain an equivalent (i.e. producing the same output value) definite integral for Clue 3 that does not involve any trigonometric functions.

(c) What is the equivalent definite integral Keisha obtains for Clue 3? *(Hint: First write* $\sin^2 x$ *in terms of* $\tan x$ *and* $\sec x$*.)* Do not solve this integral yet.

Solution: We can write $\sin^2 x = \frac{\tan^2 x}{\sec^2 x}$. Then the integral becomes

$$\int_{\arctan 2}^{\arctan 8}\frac{500}{(\sin^2 x)(\tan x + 2)}dx = \int_{\arctan 2}^{\arctan 8}\frac{500\sec^2 x}{\tan^2 x(\tan x + 2)}dx.$$

We can use u-substitution now. Let $u = \tan x$ and $du = \sec^2 x\, dx$. Then, remembering to change our bounds, we get the integral

$$\boxed{\int_2^8 \frac{500}{u^2(u+2)}du}.$$

Keisha can't finish the integral because she has not yet learned partial fraction decomposition, so she decides to approximate the transformed integral (from (c)) using MID(3). Dalon is anxious to get into the safe, and he tries entering the passcode given by Clayden's answers to the first two clues, followed by Keisha's MID(3) approximation, rounded up. The safe does not open!

(d) Without computing Keisha's MID(3) approximation of the definite integral for Clue 3 from part (c), comment on whether the passcode that Dalon tries is larger or smaller than the actual passcode.

Solution: The function $\frac{500}{u^2(u+2)}$ is concave up on the interval $[2, 8]$, and therefore MID(3) will be an underestimate of $\int_2^8 \frac{500}{u^2(u+2)}du$. Given that Dalon's attempted passcode is incorrect, it must be the case that the passcode Dalon tries is **smaller** than the actual passcode.

Clayden rushes back from Calculus class, having just learned how to do partial fraction decomposition! Dalon and Keisha are on their lunch break, and not in the store. He quickly, but correctly, solves Clue 3, and opens the safe.

(e) What is the value of the definite integral in part (c)? What is the correct passcode?

Solution: The denominator has a linear factor and a repeated linear factor. Via partial fraction decomposition, we can set up the following:

$$\int_2^8 \frac{500}{u^2(u+2)}du = \int_2^8 \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+2}du = \int_2^8 \frac{Au(u+2) + B(u+2) + C(u^2)}{u^2(u+2)}du.$$

We need

$$Au(u+2) + B(u+2) + C(u^2) = 500.$$

If $u = 0$, we can see that $2B = 500$, so $B = 250$. If $u = -2$, we can see that $C(-2)^2 = 500$, so $C = 125$. If $u = 1$, then $3A + 3 \cdot 250 + 125 = 500$ so $A = -125$. Then we have the integral

$$\int_2^8 \left(\frac{-125}{u} + \frac{250}{u^2} + \frac{125}{u+2}\right)du.$$

We compute

$$\begin{aligned}\int_2^8 \frac{-125}{u} + \frac{250}{u^2} + \frac{125}{u+2}du &= \left[-125\ln|u| - 250u^{-1} + 125\ln|u+2|\right]_2^8 \\ &= -125\ln|8| + 125\ln|2| - 250/8 + 250/2 + 125\ln|10| - 125\ln|4| \\ &= 125\ln(5/8) + 750/8 \boxed{\approx 34.9995}.\end{aligned}$$

So Clue 3 is describing the number **35**, the smallest integer larger than the result above.
The passcode is therefore **631635**.

The safe contains a bunch of small collectible items. Keisha and Dalon return from lunch and tell Clayden all about their experience with the safe. For a long day of solving integrals, Dalon makes sure Clayden and Keisha receive their fair shares of the spoils!

2. VCookies is a bakery cafe chain, known best for their vegan cookies. They are planning to add a new item to their cookie lineup, the vegan Linzer cookies. These cookies will be sold in circular trays of diameter 18 inches. Milana works as a design engineer for VCookies, and she has been tasked with designing lids for these circular cookie trays.

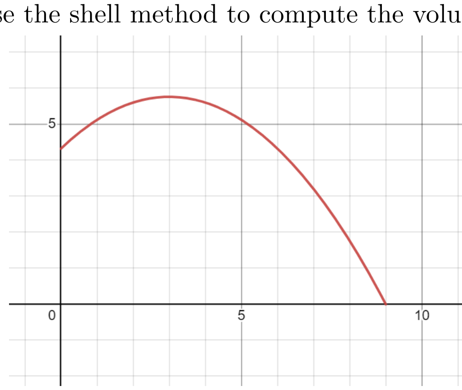
   Milana considers the aesthetics, cost of construction and ergonomics and arrives at the following three models for the cookie tray lid, that she then presents to the executive board of VCookies. For each model below, assume x and y are in inches, and that the circular cookie tray (that Milana is attempting to build a lid for) is placed perpendicular to the y-axis with a diameter on the x-axis, so that its center coincides with the origin.

   - **Model 1:** The cookie tray lid is formed by rotating the function, $f(x) = -\left(\frac{2x-6}{5}\right)^2 + \left(\frac{12}{5}\right)^2$ on the interval $0 \leq x \leq 9$, around the y-axis.
   - **Model 2:** The cookie tray lid is formed by rotating the function, $g(y) = \begin{cases} 9+y, & \text{for } 0 \leq y < 2 \\ 15-y^2, & \text{for } 2 \leq y \leq \sqrt{15} \end{cases}$ around the y-axis.
   - **Model 3:** The cookie tray lid fully encloses the cookie tray, and the entire container (the lid together with the cookie tray) has rectangular cross sections perpendicular to the tray (and the base of each cross section is perpendicular to the x-axis). The cross-sectional rectangles have height one-third of their width.

   (Note: Please feel free to use Desmos to help visualize the cookie trays, and also please include visual aids for each of the three models in your submission. You have already worked hard solving integrals by hand in problem 1, so you can use a computational device to evaluate all the integrals in this problem. Just be sure to fully write out the expression for each integral you evaluate.)

   (a) The executive board concludes its discussion and announces that, to maximize sales (i.e. amount of cookies sold), it will be approving the model with the largest volume enclosed by the lid and the cookie tray. Which model does the executive board approve?

Solution: **Model 1**: We use the shell method to compute the volume.



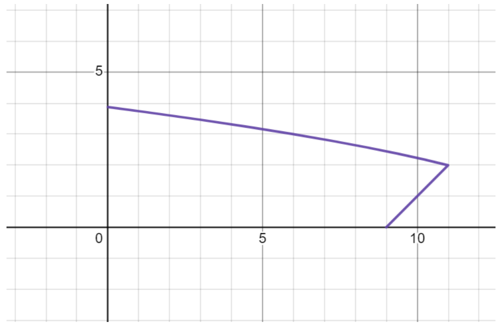
A shell has radius x, height $f(x)$ and thickness Δx. Therefore its volume is $2\pi x f(x)\Delta x$. The volume of the cookie tray+lid is given by

$$\int_0^9 2\pi x(-(\frac{2x-6}{5})^2 + (\frac{12}{5})^2)dx = 2\pi \int_0^9 (\frac{-4x^3}{25} + \frac{24x^2}{25} - \frac{36x}{25})dx + 2\pi(\frac{12}{5})^2 \int_0^9 xdx$$

$$= 2\pi(-\frac{x^4}{25} + \frac{8x^3}{25} - \frac{18x^2}{25})\Big|_0^9 + 2\pi(\frac{12}{5})^2 \frac{x^2}{2}\Big|_0^9$$

$$= \frac{1458}{5}\pi \approx \mathbf{916.09} \text{ inches}^3.$$

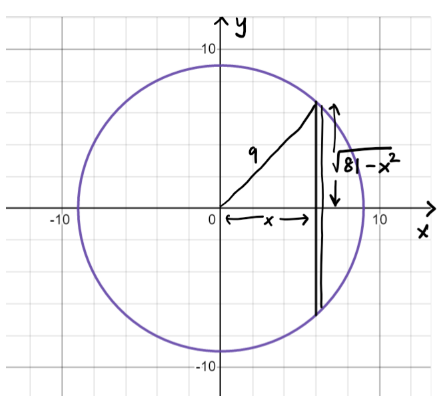
Model 2: We use the disk method with horizontal disks.

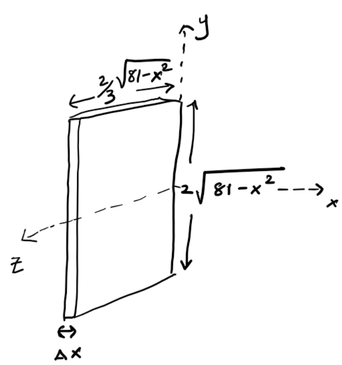


Each disk has radius $g(y)$ and thickness Δy, so the volume of a single disk is $\pi(g(y))^2\Delta y$. The volume of the cookie tray+lid is given by

$$\int_0^{\sqrt{15}} \pi(g(y))^2 dy = \pi\int_0^2 (9+y)^2 dy + \pi\int_2^{\sqrt{15}} (15-y^2)^2 dy$$
$$= \pi(81y + 9y^2 + \frac{y^3}{3})\big|_0^2 + \pi(225y - 10y^3 + \frac{y^5}{5})\big|_2^{\sqrt{15}}$$
$$\approx 289.025\pi \approx \mathbf{907.999} \text{ inches}^3.$$

Model 3: The width of a rectangular cross section perpendicular to the $x-$axis is $2\sqrt{81-x^2}$, its height is $\frac{2}{3}\sqrt{81-x^2}$, and thickness is Δx. Given below is an image of the base of the cookie tray.





The volume of such a slice is approximately $\frac{4}{3}(81-x^2)\Delta x$. The volume of the cookie tray+lid is given by

$$\frac{4}{3}\int_{-9}^{9}(81-x^2)dx = \frac{8}{3}\int_0^9 (81-x^2)dx$$
$$= \frac{8}{3}(81x - \frac{x^3}{3})\big|_0^9$$
$$= \mathbf{1296} \text{ inches}^3.$$

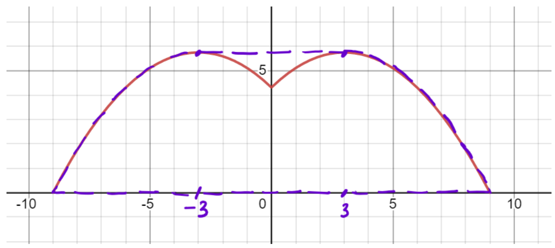
It is clear that **Model 3** has most volume, and this is the model the board approves.

(b) Milana is disappointed that the board came to such a terse decision, and attempts to pitch her models once again. She notes that the holiday season is approaching and, keeping with tradition, VCookies will wrap a ribbon, along the diameter on the x-axis, around each container (the lid plus the tray). And since ribbons are expensive, she asks that the board reconsider its choice of model for the holiday season. The board deliberates for a while, and eventually approves of Milana's notion. For the holiday season, VCookies will use the model for the lid that requires the least length of ribbon to wrap. In its updated announcement, does the executive board approve of a different model for the holiday season? *(Note: The ribbon is tied taut around the container so that if there were a trough at the top, the ribbon would simply stretch between the two peaks.)*

Solution: **Model 1**: Consider the function $h(x)$ that represents the cross-section of the lid through the $x-$axis:

$$h(x) = \begin{cases} f(x) & 0 \leq x \leq 9 \\ f(-x) & -9 \leq x \leq 0 \end{cases}$$

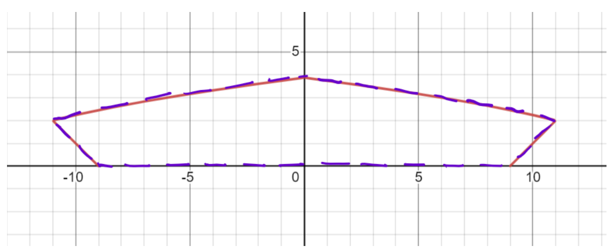
The red graph below is the graph of h:



The purple dotted line represents the taut ribbon. h has peaks at $x = -3, 3$ and so the length of the ribbon required is 18 + arc length of $h(x)$ on $[-9, -3]$ + 6 + arc length of $h(x)$ on $[3, 9]$. Due to symmetry, this length is equal to

$$24 + 2\int_3^9 \sqrt{1 + (h'(x))^2}dx = 24 + 2\int_3^9 \sqrt{1 + \left(\frac{4}{5}\right)^2 \left(\frac{2x-6}{5}\right)^2}\,dx$$
$$\approx 24 + 17.387 = \mathbf{41.387} \text{ inches.}$$

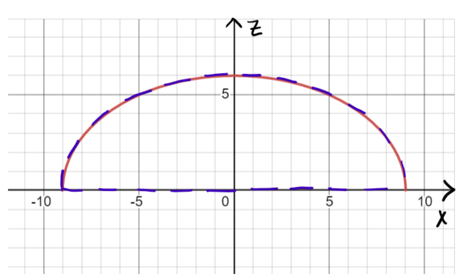
Model 2:



In this model, the length of ribbon required is 18+2(arc length of $g(y)$ on $[0, \sqrt{15}]$), which is equal to

$$18 + 2\int_0^2 \sqrt{1+1}dy + 2\int_2^{\sqrt{15}} \sqrt{1+4y^2}dy \approx 18 + 5.6569 + 22.3276 \approx \mathbf{45.985} \text{ inches.}$$

Model 3: The below image represents a cross section of the lid cut through the x-axis.



This is the graph of the function $k(x) = \frac{2}{3}\sqrt{81 - x^2}$ when $-9 \leq x \leq 9$. The length of ribbon required is 18+ arc length of $k(x)$ on $[-9, 9]$, which is equal to

$$18 + \int_{-9}^{9} \sqrt{1 + (k'(x))^2}dx = 18 + 2\int_0^9 \sqrt{1 + \frac{4x^2}{9(81 - x^2)}}dx$$
$$\approx 18 + 23.798 = \mathbf{41.798} \text{ inches.}$$

The board now approves a different model, namely **Model 1**.