Solutions to Team HW #6

1 (a) $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$ has interval of convergence $(-1,1)$

$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$ Since it's a geometric series with ratio $-x^2$, so it converges when $|-x^2| < 1$.

Substitute $\frac{x}{2}$ for x:

$\frac{1}{1+\frac{x^2}{4}} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n}}$, converges when $|\frac{x}{2}| < 1$, so interval of convergence is $(-2,2)$.

(b) $\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ has interval of convergence $(-1,1]$.

$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

Differentiate: $\frac{1}{x+1} = \sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1} = \sum_{n=0}^{\infty} (-1)^n x^n$ has interval of convergence $(-1,1)$.

(c) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ has interval of convergence $(-\infty,\infty)$

Substitute x^2 for x: $e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$ has int. of conv. $(-\infty,\infty)$

Integrate: $\int_0^x e^{t^2} dt = \sum_{n=0}^{\infty} \int_0^x \frac{t^{2n}}{n!} dt = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\cdot(n!)}$ has int. of conv. $(-\infty,\infty)$

2 (a) $1 = \int_{-\infty}^{\infty} f(t)dt = \int_0^3 w\, dt = 3w$, so $\boxed{w = \frac{1}{3}}$

(b) $f(1) = \frac{1}{3}$. This means that the probability the stick was chopped somewhere within a small interval of length Δx which contains the point 1 will be approximately $\frac{\Delta x}{3}$, regardless of the choice of the interval.

(c) $F(x) = \underbrace{\int_0^x f(t)dt = \int_0^x \frac{1}{3} dt}_{\text{if } 0 \le x \le 3} = \frac{x}{3}$, so $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x/3 & \text{if } 0 \le x \le 3 \\ 1 & \text{if } x > 3 \end{cases}$

(d) $F(\frac{1}{\pi}) = \frac{1}{3\pi} \approx 10.61\%$

This means that roughly 10.61% of the time, the left piece of the broken stick will be at most $\frac{1}{\pi}$ meters long.

3 (a) ~~Since $A(r)$ is a cdf, $\lim_{r\to\infty} A(r) = 1$ and $\lim_{r\to-\infty} A(r) = 0$, so $c=1$ and $b=0$.~~

Since the dart always hits the board, we have $A(r)=1$ for $r \ge 9$ and $A(r)=0$ for $r<0$, so $c=1$ and $b=0$ and $1 = A(9) = 81k \Rightarrow k = \frac{1}{81}$.

(b) Writing $p(t)$ for the probability density function, we know that $p(t)=0$ if $t<0$ or $t>9$, and that for $0\le r\le 9$ we have $\frac{r^2}{81}=A(r)=\int_0^r p(t)\,dt$. Differentiate, using FTC, to get $\frac{2r}{81}=p(r)$ for $0\le r\le 9$.

(c) (i) $A(3)-A(1)=\frac{9}{81}-\frac{1}{81}=\frac{8}{81}$

(ii) $1-A(7)=1-\frac{49}{81}=\frac{32}{81}$

(d) $\text{Mean}=\int_{-\infty}^{\infty} t\,p(t)\,dt=\int_0^9 t\cdot\frac{2t}{81}\,dt=\int_0^9\frac{2}{81}t^2\,dt=\frac{2}{81}\frac{t^3}{3}\Big]_0^9=6$ inches

Median $=T$ such that $A(T)=\frac{1}{2}$, so $\frac{1}{2}=\frac{T^2}{81}\Rightarrow T^2=\frac{81}{2}\Rightarrow T=\frac{9}{\sqrt{2}}\approx 6.36$ inches

Meaning: when you throw darts, the average location is 6 inches from the center, and half the time the location is at most 6.36 inches from the center.

4 (a) No – the ocean is huge.

(b) Take logs: $\ln(P)=N_c\cdot\ln\left(1-\frac{N_c}{N_w}\right)$.

Since $\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ for $-1<x\le 1$, substituting $-\frac{N_c}{N_w}$ for x yields $\ln\left(1-\frac{N_c}{N_w}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-1)^n\left(\frac{N_c}{N_w}\right)^n}{n}=-\sum_{n=1}^{\infty}\frac{\left(\frac{N_c}{N_w}\right)^n}{n}$ when $-1<-\frac{N_c}{N_w}\le 1$, which is true in our situation since $0<N_c<N_w$.

Thus $\ln(P)=-N_c\cdot\sum_{n=1}^{\infty}\frac{\left(\frac{N_c}{N_w}\right)^n}{n}$.

(c) $\ln(P)\approx -N_c\cdot\sum_{n=1}^{3}\frac{\left(\frac{N_c}{N_w}\right)^n}{n}\approx -1546.23$

(d) $P\approx e^{-1546.23}\approx 3.03\times 10^{-672}$

$1-P\approx 1$. So part (a) above was wrong!

5 (a) "Guess" $f(x)=\frac{e^x+e^{-x}}{2}$. Check: $f(0)=1$, $f'(x)=\frac{e^x-e^{-x}}{2}$, $\sqrt{1+(f'(x))^2}=\left(\frac{e^x+e^{-x}}{2}\right)$ $=f''(x)$

(b) $\frac{dy}{dx}=y'=-x\left(y-\frac{1}{2}\right)\Rightarrow\frac{dy}{y-\frac{1}{2}}=-x\,dx\Rightarrow\ln\left|y-\tfrac{1}{2}\right|=\frac{-x^2}{2}+C\Rightarrow\left|y-\tfrac{1}{2}\right|=e^{-\frac{x^2}{2}+C}$

$\Rightarrow y-\frac{1}{2}=\pm e^C\cdot e^{-x^2/2}=D\cdot e^{-x^2/2}$, so $f(x)=\frac{1}{2}+D\cdot e^{-x^2/2}$, and $f(0)=1$ implies $D=\frac{1}{2}$, so $f(x)=\frac{1}{2}+\frac{1}{2}e^{-x^2/2}$

(c)