MATH 116 — PRACTICE FOR EXAM 1

Generated February 8, 2023

NAME: SOLUTIONS

INSTRUCTOR: __________ SECTION NUMBER: ______

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1. This exam has 5 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2017	3	8	spring	11	
Fall 2021	1	8	soup	10	
Winter 2021	2	3	water rocket	7	
Fall 2017	1	9	chain	8	
Fall 2019	2	6	bucket	8	
Total				44	

Recommended time (based on points): 43 minutes

8. [11 points] Imagine that a one pound ball is attached to a spring. This ball is allowed to move forward and backward on a table, but not up and down (or side to side). When the spring is not stretched at all, we say that the ball is at its *starting position.* Let x be the displacement of the ball from its starting position in the forward/backward direction. (The value of x is positive if the ball has moved forward from its starting position and negative if the ball has moved backward from its starting position.)

a. [4 points] Let $F(x)$ be the magnitude of the force, measured in pounds, that the spring exerts on the ball when the ball has been pulled x feet from its starting position. Suppose $F(x) = 5x$.

i. Which of the following best estimates the work, in foot-pounds, needed to move the ball a very small distance Δx feet forward from a position x? *Circle* ONE *choice.*

I. 5 II. $5x$ III. $2.5x^2$ IV. $5\Delta x$ [✓ V. $5x\Delta x$] VI. $2.5x^2\Delta x$

ii. Use your answer to part i. to write an expression involving one or more integrals that gives the total work needed to move the ball from its starting position forward a distance of one half of one foot (i.e. 6 inches). Then compute the value of your integral (either by hand or using your calculator). Include units on your answer.

$\int_0^{.5} 5x\,dx = \frac{5}{2}x^2\Big|_0^{.5} = \frac{5}{2}\left(\frac{1}{2}\right)^2$

Answer: Integral Expression: $\boxed{\int_0^{.5} 5x\,dx}$

Numerical Answer (with units): $\boxed{5/8 \text{ Foot pounds}}$

b. [4 points] After stretching the spring as described above, you release it from a starting position of $x = 1/2$. The ball oscillates backwards and forwards (in the x-direction), and its position $x = x(t)$ satisfies the differential equation $x'' + 5x = 0$. Note that $x'' = \frac{d^2x}{dt^2}$. For what values of A, B, and k will the function

$$x(t) = A\sin(kt) + B\cos(kt)$$

be a solution to the differential equation $x'' + 5x = 0$ with the initial conditions $x(0) = 1/2$ and $x'(0) = 0$?

$x'(t) = kA\cos(kt) - kB\sin(kt)$

$x''(t) = -k^2A\sin(kt) - k^2B\cos(kt)$

Since $x'' = -k^2x$, $k^2 = 5$

$\frac{1}{2} = x(0) = B$

$0 = x'(0) = kA$

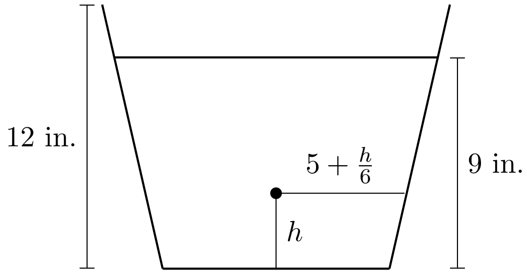
Answer: $A = \boxed{0}$ and $B = \boxed{1/2}$ and $k = \boxed{\pm\sqrt{5}}$

c. [3 points] Using the particular solution that you found in part **b**, find the first time $t > 0$ when the ball reaches the position $x = 0$.

$x(t) = \frac{1}{2}\cos(\sqrt{5}t) = 0 \Rightarrow \sqrt{5}t = \frac{\pi}{2} + m\pi$

Answer: $t = \boxed{\pi/2\sqrt{5}}$

8. [10 points] Frank, an aspiring chef and looking to impress his extended family, makes a big pot of tomato soup that he brings to his family reunion potluck. The pot is 12 inches tall with **circular cross sections** parallel to the bottom of the pot. The cross section h inches from the bottom of the pot has radius $5 + \frac{h}{6}$ inches for $0 \leq h \leq 12$. Unfortunately, his least favorite cousin Barrie brought a better tomato soup to the potluck. Almost no one ate Frank's soup and after the potluck, the pot still had soup up to 9 inches above the bottom of the pot. Frank saved the rest of the soup for himself and now he has to get the remaining soup out of the pot.



a. [3 points] Write an expression for the volume of a thin horizontal slice of soup at height h from the bottom of the pot and thickness Δh. Make sure to include units.

Solution: Taking our horizontal slice the volume ΔV can be expressed using the fact that the radius $r = 5 + \frac{h}{6}$:

$$\Delta V = \pi r^2 \Delta h = \pi(5 + \frac{h}{6})^2 \Delta h \text{ in.}^3$$

b. [4 points] By the end of the potluck, the soup has settled into an uneven density. The density of the soup at height h above the bottom of the pot is $.05(1+h)$ pounds/in^3. Write an expression for the amount of work in pound-inches required to get a thin horizontal slice of soup at height h above the bottom of the pot and thickness Δh to the top of the pot.

Solution: The thin horizontal slice at height h above the bottom of the pot and thickness Δh must be raised $12 - h$ inches. It has density $.05(1 + h)$ pounds/in^3. So, the work ΔW done in raising the horizontal slice is

$$\Delta W = .05(1 + h)\pi(5 + \frac{h}{6})^2(12 - h)\Delta h.$$

c. [3 points] Write a definite integral that represents the total amount of work in pound-inches required to get all the soup that was left in the pot after the potluck out of the pot. Do not evaluate the integral.

Solution: The pot is 9 in. full, so the total work done in raising the soup is

$$\int_0^9 .05\pi(1 + h)(5 + \frac{h}{6})^2(12 - h)dh.$$

3. [7 points] Emily is a physics teacher. She is demonstrating the physics of rocket-launching with a water rocket in her class today. The water rocket weighs 3 lbs on the ground, and Emily launches it straight up to 10 ft above the ground. During the launch, the rocket's weight decreases at constant rate (in lbs/ft) as the water is ejected from the rocket. When it reaches 10 ft above the ground, the rocket weighs 1 lb.

a. [3 points] Calculate the weight of the rocket when it is a height h ft above the ground. Include units.

Solution: Since the rocket's weight decreases at a constant rate in lb / ft, the rocket's weight is a linear function of its height above the ground, i.e. h.
At $h = 0$, the weight is 3lb. At $h = 10$, the weight is 1lb. Hence the slope of weight in terms of height is

$$\frac{1-3}{10-0} = \frac{-1}{5}.$$

By using point-slope form with $h = 0$ and weight $=$ 3lb, we have that

$$(\text{Weight at height } h) - 3 = \frac{-1}{5}(h-0),$$

$$\text{Weight at height } h = 3 - \frac{h}{5} \text{ lb}.$$

b. [4 points] Write an expression involving integrals for the total work required to propel the rocket from the ground to a height of 10 feet above the ground (as described above). Do not evaluate any integrals in your expression. Include units.

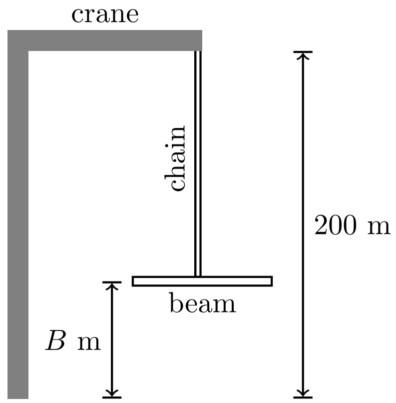
Solution: Work to lift the rocket from h ft above ground to $h + \Delta h$ ft above ground is

$$(\text{Weight at height } h)(\text{lb}) \cdot (\text{distance travelling})(\text{ft}) = (3 - \frac{h}{5})\Delta h(\text{ftlb}).$$

Note that there is no $g = 9.8$, since lb is a unit for force. Hence total work is

$$\int_0^{10} (3 - \frac{h}{5})\, dh \text{ ftlb}.$$

9. [8 points]
During the construction of a skyscraper, a 200 meter tall crane lifts a steel beam from the ground to a height of 175 meters. The steel beam has a mass of 50 kilograms. The crane has a chain that is also made of steel, and the chain has a mass of 15 kilograms per meter. The total length of the chain is 200 meters, but as the beam is lifted, the crane no longer needs to lift any of the chain that has already been "reeled in", i.e. has already reached the top of the crane.



For this problem, you may assume the acceleration due to gravity is $g = 9.8$ m/s^2 .

a. Write an expression in terms of B that gives the total mass, in kilograms, of the steel beam together with the chain that has not yet been reeled in at the moment that the steel beam is B meters above the ground.

Solution: If the steel beam is B meters above the ground, that means that a length of B meters of chain has been reeled in, so $(200-B)$ meters of chain remains. The mass of this remaining chain is (15 kilograms per meter) $\cdot$ $(200 - B$ meters$) = 3000 - 15B$ kilograms. We add to this the mass of the steel beam to find a total mass of

$$\text{Mass} = 50 + 3000 - 15B \text{ kilograms.}$$

b. Assuming ΔB is very small but positive, write an expression in terms of B that approximates the work done by the crane in lifting the steel beam up ΔB meters starting from a height of B meters above the ground. Assume that the weight of the chain being lifted is constant over this very short distance. Include units.

Solution: The force due to gravity is the weight. At the moment the steel beam is B meters above the ground, the weight of the beam together with the chain that has not yet been reeled in is

$$\text{Weight} = (\text{mass})(g) = (3050 - 15B \text{ kilograms})(9.8 \text{ m/s}^2) = 29890 - 147B \text{ Newtons.}$$

The work to lift the steel beam over the small distance ΔB is then approximately

$$(\text{Force}) \cdot \Delta B = (29890 - 147B)\Delta B \text{ Joules.}$$

c. Write, but do **not** evaluate, an expression involving one or more integrals that gives the total work that must be done by the crane in order to lift the steel beam from the ground to a height of 175 meters. Include units.

Solution: Using our answer from part **b.** above, summing over the entire path of the beam, and taking the limit as ΔB approaches 0, we find that the total work is

$$\text{Work} = \int_0^{175} g(3050 - 15B)\, dB = \int_0^{175} (29890 - 147B)\, dB \text{ Joules.}$$

6. [8 points] Derivative Girl lifts a bucket of water at a constant velocity from the ground up to a platform 50 meters above the ground. The bucket and water start at a total mass of 20 kg, but while it is being lifted, a total of 3 kg of water drips out at a steady rate through a hole in the bottom of the bucket.

For this problem, you may assume that acceleration due to gravity is $g = 9.8$ m/s^2.

a. [2 points] Give an expression giving the mass of the bucket and water when the bucket is h meters above ground. Include units.

Answer: Mass of water = $20 - \frac{3}{50}h$ **kg**

b. [3 points] Suppose Δh is small. Write an expression (not involving integrals) that approximates the work required to lift the bucket from a height of h meters above the ground to a height of $h + \Delta h$ meters above the ground. Include units.

Answer: Work ≈ $9.8(20 - \frac{3}{50}h)\Delta h$ **joules**

c. [3 points] Write, but do not evaluate, an integral that gives the work required to lift the bucket from the ground to the platform. Include units.

Answer: $\int_0^{50} 9.8(20 - \frac{3}{50}h)\, dh$ **joules**