WRITING INEQUALITIES

1. Standard functions to compare against

The following are some common integrals one uses for comparison test:

- The integral $\int_0^\infty e^{ax}\,dx$ converges if $a < 0$, and diverges if $a \geqslant 0$.
- The integral $\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges if $p \leqslant 1$
- The integral $\int_0^1 \frac{1}{x^p}\,dx$ converges if $p < 1$ and diverges if $p \geqslant 1$.

Note:

- The interval of integration in the two p-tests.
- Both the integrals in the p-test diverge when $p = 1$.

Because these functions are commonly used in applying the comparison test for integrals, you may find yourself trying to write inequalities involving these functions frequently.

2. Standard range restrictions

The following are useful and commonly-used inequalities:

$$\begin{aligned} -1 &\leqslant \sin(x) \leqslant 1 \\ -1 &\leqslant \cos(x) \leqslant 1 \\ -\frac{\pi}{2} &\leqslant \arcsin(x) \leqslant \frac{\pi}{2} \\ 0 &\leqslant \arccos(x) \leqslant \pi \\ -\frac{\pi}{2} &\leqslant \arctan(x) \leqslant \frac{\pi}{2} \end{aligned}$$

But you should know how to use these to get other inequalities. Try your hand at proving the following:

(1) Show that $2 \leqslant 2 + \sin^2(x) \leqslant 3$ for all x.
(2) Show that $\frac{1}{t+\pi} \leqslant \frac{1}{t + \arccos(t)} \leqslant \frac{1}{t}$ for all $t > 0$.

3. Techniques for writing inequalities

Here are some recipes to come up with inequalities. Remember that you are writing inequalities because you want to get to simpler function in the end than the one you are starting with. Usually, this is because the simpler function you want to get is a) one of the above "common functions" used in the comparison test, or b) easier to integrate directly.

- **Standard range restrictions.** It is frequently useful to use the standard range restrictions listed above when writing inequalities.
  For example, since $-1 \leqslant \sin(x) \leqslant 1$, we know that $\dfrac{1+\sin(x)}{x^2} \leq \dfrac{2}{x^2}$.
- **Making the numerator bigger/smaller.** It may be useful for you to make the numerator bigger/smaller. Making the numerator bigger gives you an upper bound, and making it smaller gives you a lower bound.
  For example, on $(0, \infty)$: Given $\dfrac{x^2+4}{x^3}$, we can "forget" the $+4$ to get the inequality $\dfrac{x^2+4}{x^3} \geqslant \dfrac{x^2}{x^3} = \dfrac{1}{x}$.
- **Making the denominator bigger/smaller.** It may be useful for you to make the denominator bigger/smaller. Making the denominator bigger makes the fraction smaller, giving you a lower bound, and making the denominator smaller gives you an upper bound.
  For example, on $(5, \infty)$: Given $\dfrac{x^2}{x^4+2x^2}$, we can "forget" the $+2x^2$ to get the inequality $\dfrac{x^2}{x^4+2x^2} \leqslant \dfrac{x^2}{x^4} = \dfrac{1}{x^2}$.
- **Leading term/lagging term analysis.** When deciding to add or drop terms while making the numerator/denominator bigger or smaller, you want to decide which term to make bigger or which term to drop by considering each term's growth rate.
  In the previous example, for instance, we dropped the $2x^2$ rather than the x^4 in the denominator, since as $x \to \infty$, x^4 dominates. In the first example, we dropped the $5\sqrt{x}$, since as $x \to \infty$, the x^2 term dominates.
- **Increasing/decreasing functions.** If you know that a function is increasing for $x \geqslant a$, then you get the inequality $f(x) \geqslant f(a)$. Similarly, once you know that a function is decreasing for $x \geqslant a$, then you get the inequality $f(x) \leqslant f(a)$.
  For example, e^{-x} is a decreasing function. Therefore, $e^{-x} \leq e^{-0} = 1$ for all $x \geq 0$. Therefore, for $x \geq 0$, $\dfrac{e^{-x}}{x^2+x+1} \leq \dfrac{1}{x^2+x+1}$.
- **Concavity.** Sometimes concavity of the function can help produce inequalities with linear functions. If a function $f(x)$ is concave up (resp. concave down) on an interval $[a, b]$, then the secant line between the end points $(a, f(a))$ and $(b, f(b))$ lies above (resp. below) the graph $f(x)$ on the interval $[a, b]$. Also, if $f(x)$ is concave up (resp. concave down) on (a, b), then $f(x)$ lies above (resp. below) its tangent line (except at the point of contact between the graph and the tangent line).

For example, consider $\sin(x)$ on the interval $[0, \pi/2]$. On this interval, $\sin(x)$ is concave down. Therefore it lies above the secant line between $(0,0)$ and $(\pi/2, 1)$, which is the line $y = \frac{2}{\pi}x$, and so $\sin(x) \geq \frac{2}{\pi}x$ on $[0, \pi/2]$.

For an example concerning the tangent line, we consider $\ln(x)$ on the interval $(0, \infty)$. The function $\ln(x)$ is concave down, and therefore lies below the tangent line to $\ln(x)$ at $x = 1$, which is the line $y = x - 1$. Therefore, we have $\ln(x) \leq x - 1$.

In the context of Improper Integrals

- **Guessing behavior.** When faced with determining convergence or divergence of an improper integral, the first thing to do is make an educated guess about the behavior of the function and its integral – will the integral converge or diverge? Remember that *if you want to show convergence, you'd need a convergent upper bound, and if you want to show divergence, you need a divergent lower bound.* To guess the shape of the inequality you need (i.e., an upper bound or a lower bound?), you can use the growth rate of the function in the integrand and use it to arrive at a guess of whether you think the integral will converge or diverge.
  For example, if we are given the integral $\int_1^\infty \frac{1}{x^2 + 5\sqrt{x}}\,dx$, note that as x tends to infinity, x^2 dominates $\sqrt{x}$, and therefore the denominator "behaves like x^2 for large x-values". Therefore we guess that $\int_1^\infty \frac{1}{x^2 + 5\sqrt{x}}\,dx$ behaves like $\int_1^\infty \frac{1}{x^2}\,dx$, which converges. **Note:** The fact that x^2 dominated $5\sqrt{x}$ depended on the fact that the interval considered here was $(1, \infty)$. If we were on $(0, 1)$, $5\sqrt{x}$ would dominate x^2.
- **Eventual behavior matters.** When trying to show a certain convergence/divergence behavior, remember a) you are allowed to introduce your own (definite) constants, and b) it is frequently enough to have an inequality for all $x \geq x_0$ for some number x_0.
  For example, consider the integral $\int_{1/2}^\infty \frac{1}{x}e^{-x}\,dx$. The integrand is *not* less than e^{-x} on the full interval of integration. But it is less than e^{-x} for all $x \geq 1$. Since $\int_{1/2}^1 \frac{1}{x}e^{-x}\,dx$ is not improper, the behavior on $[1, \infty)$ is the main ingredient.